Linear and Dynamic Programming in Markov Chains*

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Some essential elements of the Markov chain theory are reviewed, along with programming of economic models which incorporate Markovian matrices and whose objective function is the maximization of the present value of an infinite stream of income. The linear programming solution to these models is presented and compared to the dynamic programming solution. Several properties of the solution are analyzed and it is shown that the elements of the simplex tableau contain information relevant to the understanding of the programmed system. It is also shown that the model can be extended to cover, among other elements, multiprocess enterprises and the realistic cases of programming in the face of probable deterioration of the productive capacity of the system or its total destruction.

RECENTLY there has been growing interest in programming of economic processes which can be formulated as Markov chain models. One of the pioneering works in this field is Howard's *Dynamic Programming and Markov Processes* [6], which paved the way for a series of interesting applications. Programming techniques applied to these problems had originally been the dynamic, and more recently, the linear programming approach. Practically, a computer program to execute the dynamic programming calculation is simpler to prepare than one for the linear programming procedure. On the other hand, linear programming routines are readily available and allow great flexibility, as in parametric programming and sensitivity analysis. These features can be introduced into dynamic programming routines, although at an increasing cost. In this article we will show the lines of similarity between the two techniques and investigate some possible extensions and applications.

A finite Markov chain is a statistical model useful in describing various economic phenomena.[1] In this model, we envisage a process which is in a certain *state* i, where $i=1, 2, \cdots, n$ (n finite), in a particular *period* or *stage*, and is transformed in the next period to a state j ($j=i$ is permissible). The chain is described by an n-order *transition*, or *Markov* matrix, whose elements p_{ij} are the probabilities that the process will go from state i to state j. These probabilities are independent of the past history of the process.

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[1] For a rigorous and complete treatment of Markov chains see Kemeny and Snell [8].

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For example, let us consider a field whose state is defined by the level of humidity of the soil (measured in discrete units). The field may be transformed from one state to another with certain probabilities, depending on crop and weather conditions [1]. Additional illustrations might be a system of pieces of equipment whose failures are a stochastic process [2], or a warehouse where the state is given by the level of inventory [3, 4, 7].

In economic processes, with every state is associated a reward—or cost—for example, yield of the field, repair of machine, profits from sales of items out of inventory. The interesting cases are those in which the transition probabilities can be affected by *action.* A *policy* will then be the rule which dictates an action to be taken in every state. An *optimal* policy will be the policy under which total expected income from the process is maximized. In this framework, programming is the choice of an optimal policy from a given set of alternatives. The choice can be made efficiently by either dynamic or linear programming methods. We will investigate the relations between the two methods and interpret the results of the linear programming calculations. We hope to show not only that linear programming is applicable in this context, as has already been shown [3, 4, 7, 9], but also that its interpretation throws light on the "anatomy" of the system and clarifies understanding of its properties.

In order to simplify the discussion, we will make several assumptions to be relaxed later in the article. First, we assume regular Markov chains, that is, any state is probable far enough in the future. Also we explicitly assume that the transition matrices are not decomposable, that is, that the process cannot be split into two or more isolated chains. We further assume that a series of processes has a unique maximum present value. The discussion is limited to processes of indefinite duration—that is, an infinite economic horizon is assumed.

Income Streams

We start the discussion by noting the mathematical equivalence of three analogous income streams[2] and naming these parallel cases for future reference. As usual, an income stream is defined by its annuity—a_t in period t.

The discounting case

Assume that a process yielding income lasts forever and that $a_t = a_0$ for all t. Let r be the appropriate rate of interest and $\alpha = 1/(1+r)$ be the discounting factor. Then the worth of the source of income—its present value—is

$$z_\alpha = \sum_{t=0}^{\infty} \alpha^t a_0 = a_0/(1-\alpha), \tag{1}$$

since $0 < \alpha < 1$.

[2] In the present context, the analogy was first introduced by D'Epenoux [4].

The deterioration case

Assume now that income from the source is not constant, but deteriorates at the rate β, where $\beta = a_{t+1}/a_t$ and $0<\beta<1$ (radioactive decay). Then the present *not discounted* value of the source of income is

(2) $$z_\beta = \sum_{t=0}^{\infty} a_t = \sum_{t=0}^{\infty} \beta^t a_0 = a_0/(1-\beta).$$

The breakdown case

In this third case, consider a constant annuity, a_0, as long as the source of income exists. There is, however, a constant probability $1-\gamma$, at every period t, that the source will be destroyed before the coming of the next period. Hence, γ is the probability of survival. Here expected worth of the income stream (not discounted) is

(3) $$z_\gamma = \sum_{t=0}^{\infty} \gamma^t a_0 = a_0/(1-\gamma).$$

These three cases are mathematically equivalent. Of course, they could be consolidated into one general case which would constitute a mixture of the three. In the course of our discussion, we shall make use of the analogy of the separate cases, as well as of the mixed case.

It will also be useful if we note that the previous equations can be rewritten in a slightly different form. Instead of (1), for example, write the recurrence relation

(1′) $$z_\alpha = a_0 + \alpha \sum_{t=0}^{\infty} \alpha^t a_0$$
$$= a_0 + \alpha z_\alpha.$$

We have named this new form the *two-steps form* of (1). It emphasizes that the present value of the infinite income stream is composed of an immediate annuity, plus the present value of the same income stream started one period later. Similar forms and interpretations can be given to (2) and (3).

Markov Chains in Economic Systems

Consider a Markov chain with an n-order transition matrix $P(n \times n) = [p_{ij}]$. Since the p_{ij} elements are probabilities,

(4) $$\sum_j p_{ij} = 1 \qquad (i = 1, 2, \cdots n).$$

Let the current state of the process—state i—be denoted by a *state vec-*

tor[3] E_i $(1\times n)$. E_i is the unit row vector with the unity in position i. Given a state vector E_i, the vector E_iP is the probability vector for the states of the process in the succeeding stage. In the stage after that, the probabilities will be $(E_iP)P=E_iP^2$. In general, the probabilities for the tth period constitute the vector E_iP^t. Also, let a rewards row vector $C(1\times n)=[c_i]$ associate an immediate reward[4] with every state i. The present value of the next period's reward is, therefore, $\alpha E_iPC'$. Thus, if the process continues indefinitely, the expected present value of all future incomes—the worth of the process currently in state i—is

$$z_i = \sum_{t=0}^{\infty} E_i(\alpha P)^t C' = E_i(I - \alpha P)^{-1}C' \tag{5}$$

where α is, as previously, the discounting factor.

Utilizing scalar notation, we may introduce the two-steps form of (5):

$$z_i = c_i + \alpha \sum_{j=1}^{n} p_{ij} z_j. \tag{5'}$$

Starting from a state i, the worth of the process is the immediate reward c_i, plus the expected worths of the states of the next stage, discounted one period.

To consider all starting states, we replace E_i by the unit matrix I and write

$$Z' = I(I - \alpha P)^{-1}C' = (I - \alpha P)^{-1}C', \tag{6}$$

where Z is the $(1\times n)$ vector whose elements are the z_i values of (5).

In terms of the previous section, the case presented here is the discounting case, within the framework of the Markovian model. We shall now make use of the analogy to the breakdown case; this will link us directly to the general theory of Markov chains and provide us with convenient terminology and greater insight. Toward this end, consider a process with a transition matrix T, of the order $n+1$, which can be partitioned:

$$T = \begin{bmatrix} Q & H' \\ 0 & 1 \end{bmatrix}.$$

In T, $H(1\times n)$ is a probability vector, 0 a zero vector, and 1 a scalar. The Markov chain defined by T consists of two sets of states: one, *transient*, with the n states in Q, and one, the $(n+1)$ state—*ergodic*. Once the process

[3] E_i can be regarded as a particular case of a *state probability vector*.

[4] The assumption in the text is that the reward is associated with the occupation of the state. It is not difficult to incorporate the alternative assumption that the reward is due to a particular transition from state i to state j [7, p. 460].

reaches the ergodic state, it will be *absorbed* there and will never re-enter any of the transient states. The elements of H are, therefore, the probabilities that the process would be transformed from each of the transient states into the ergodic state. Q is the transition matrix of the transient states.

Associated with every transient set—with every matrix Q—is a *fundamental* square matrix, $V=[v_{ij}]$.

$$V = (I - Q)^{-1} = \sum_{t=0}^{\infty} Q^t. \tag{7}$$

The elements v_{ij} indicate the expected number of times that a process, currently in state i, will be in state j before being absorbed in the ergodic state (including the current stage in the count of v_{ii}). To complete the analogy, let every transient state i carry a reward c_i, and the ergodic state represent total breakdown of the system—zero income. Total expected income (*not discounted*) for a process starting in state i, is

$$\begin{aligned} z_i &= \sum_{t=0}^{\infty} E_i Q^t C' \\ &= E_i(I - Q)^{-1}C'. \end{aligned} \tag{8}$$

By defining Q of (7) and (8) as $Q=\alpha P$, we return to the discounting case and may treat the matrix αP as if it were the transient part of a Markov process. Here we shall name the v_{ij} elements of $V=(I-\alpha P)^{-1}$, the *expected discounted* number of times that a process, currently in state i, will be in state j. These numbers are finite, while physically the process will continue for an infinite duration.

Since P is a transition matrix, the sum of every row of αP is α (see equation 4), and therefore all elements of the corresponding H vector are $1-\alpha$, which is also the sum of all rows in the matrix $I-\alpha P$. Hence, total discounted number of stages in any state, starting from state i, is by (A.2) in the Appendix

$$\sum_{j=1}^{n} v_{ij} = 1/(1 - \alpha) \qquad (i = 1, 2, \cdots, n). \tag{9}$$

We can interpret this result, again utilizing the analogy to the breakdown case, as follows: $1-\alpha$ is the probability of breakdown of the system in any stage; therefore, α is the probability of survival. Hence, the total expected number of stages before breakdown will be

$$\sum_{t=0}^{\infty} \alpha^t = 1/(1 - \alpha). \tag{10}$$

The interpretation we gave to the elements of the fundamental matrix V permits the rewriting of (8) as

$$z_i = \sum_{j=1}^{n} v_{ij}c_j \qquad (i = 1, 2, \cdots, n), \tag{8'}$$

which can easily be verified algebraically and interpreted economically.

Programming will be meaningful in those cases in which a certain process can be chosen from several alternatives. Instead of enumerating all possible transition matrices, we consider an *expanded* matrix $R\ (m\times n)=[p_{ij}^{d(i)}]$, which consists of k_i different probability rows for every state i, $m=\sum_{i=1}^{n} k_i$. The superscript $d(i)$ indicates an *action* to take in state i where $d(i)=1, 2, \cdots, k_i$. Generally, we shall eliminate, for brevity, the index i of $d(i)$ and write p_{ij}^{d}. The action indicated by the superscript will affect the transition probabilities (probabilities of failure of equipment, for example, can be affected by actions of maintenance). The immediate reward of the state i is also affected by the action; for example, cost of action is deducted from the gross value of the reward. Thus, the vector C is also expanded and its elements are now c_i^d. An expanded probability matrix R of the dimension 6×2, with the corresponding immediate rewards vector C, is given in Table 1. Thus, in the table, if in state 1 action a_1 is taken, $d(1)=1$, the transition probabilities are $p_{11}^1=0.20$, $p_{12}^1=0.80$ and the expected immediate reward is $c_1^1=\$5.00$.

Table 1. An expanded transition matrix with rewards

State	Actions[a]	Probabilities of transition (Matrix R)		Immediate rewards (Vector C')
		to state 1	to state 2	
State 1	a_1	0.20	0.80	\$5.00
	a_2	0.00	1.00	4.50
	a_3	1.00	0.00	0.00
State 2	b_1	0.60	0.40	\$2.00
	b_2	0.40	0.60	2.30
	b_3	0.00	1.00	0.00

[a] Actions are listed by names. For example, a_1 is the name of the action in state 1 for which $d(1)=1$.

The Markov process will be determined when a decision vector $D(1\times n)$ is chosen, designating a $d(i)$ value for every i, that is, specifying a policy—an action to take in every possible state.[5] By deciding on a D, one chooses a particular transition matrix P, out of R, for the process at hand and a corresponding vector C of immediate rewards.

Programming for maximal expected income can be performed by the budgeting method—by listing all possible P square matrices out of R, calculating, by (5), expected worth of each, and selecting the one with the

[5] We shall regard the vector D, interchangeably, as either the vector consisting of the indices $d(i)$ or of the names of the actions a_1, b_2, etc.

highest z_i. This might be extremely laborious. Instead, dynamic or linear programming methods may be applied.

Dynamic Programming

In this section we will follow Hadley [7, pp. 454–460], who also provides the proofs for the procedure described here.

To select an optimal decision vector D by the dynamic programming method, start from an arbitrary D, call it $D(1)$, thus selecting a corresponding matrix $P(1)$ and a vector $C(1)$. Now calculate a vector $Z(1)$ of expected present values for all starting states.

$$(11) \qquad \begin{aligned} Z(1)' &= [I - \alpha P(1)]^{-1} C(1)' \\ &= C(1)' + \alpha P(1) Z(1)'. \end{aligned}$$

The last line—the two-steps form of (11)—is the matrix form of (5′).

Next, check whether $D(1)$ is optimal. This is done by the following recurrence procedure: define a *test policy* to be the policy $D(1)$ for all future stages but not necessarily for the current one. For the current stage, the test policy associates an alternative action $d(i)$—not necessarily in $D(1)$—with state i. Now evaluate

$$(12) \qquad z_i = \max_d \left[c_i^d + \alpha \sum_{j=1}^{n} p_{ij}^d z_j(1) \right] \qquad (i = 1, 2, \cdots, n).$$

A new decision vector $D(2)$ emerges, consisting, for every i, of the $d(i)^*$ element that maximizes the expression in (12). If $D(1)$ is an optimal policy, then $D(2) = D(1)$. If not, calculate

$$(13) \qquad Z(2)' = [I - \alpha P(2)]^{-1} C(2)',$$

and repeat (12) and (13) until $D(k) = D(k-1) = D^*$.[6] D^* is the optimal policy which maximizes present value of expected income from the process.

In this procedure, all possible starting states are considered. Thus, D^* is invariant under different starting states—the set of optimal actions to take in every possible state is independent of the current state of the process.

Linear Programming

Our linear programming problem [5] will be

$$(14) \qquad \begin{cases} a.\ \max C\Pi' \\ \quad \text{subject to} \\ b.\ M\Pi' = E_i' \\ c.\ \Pi \geq 0. \end{cases}$$

[6] The optimal policy need not be unique; several D vectors might lead to the same maximal present value. It is, however, not difficult to protect the computer program against cycling.

In (14), C is the expanded immediate rewards vector; Π is the solution vector to the linear programming problem; E_i is, as previously, the unit state vector with unity in position i. The matrix $M(n\times m)$ is constructed of the expanded transition matrix R by first expanding a unit matrix to a matrix $J(m\times n)$, which consists of k_i identical E_i unit row vectors for every i, and then

$$M = (J - \alpha R)'. \tag{15}$$

The matrices J, R, and M, for a problem with two states and two actions in each state, are illustrated below.

$$J = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \alpha R = \begin{bmatrix} \alpha p_{11}{}^1 & \alpha p_{12}{}^1 \\ \alpha p_{11}{}^2 & \alpha p_{12}{}^2 \\ \alpha p_{21}{}^1 & \alpha p_{22}{}^1 \\ \alpha p_{21}{}^2 & \alpha p_{22}{}^2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1-\alpha p_{11}{}^1 & 1-\alpha p_{11}{}^2 & -\alpha p_{21}{}^1 & -\alpha p_{21}{}^2 \\ -\alpha p_{12}{}^1 & -\alpha p_{12}{}^2 & 1-\alpha p_{22}{}^1 & 1-\alpha p_{22}{}^2 \end{bmatrix}$$

Table 2 is the simplex table for the example of Table 1. The matrix M constitutes the bulk of the first section—the input–output coefficients—to which a unit matrix of slack variables (artificial activities) was added. The assumption in the table is that the process is started in state 1.

We shall now show that the solution to the linear programming problem (14), like the dynamic programming solution, will select a policy that will maximize expected present value of income from the process at hand.

Following the usual linear programming convention, we add slack variables and partition the vectors Π and C and the matrix M:

$$\Pi = [\Pi_s \quad \Pi_o \quad \Pi_1], \qquad C = [C_s \quad C_o \quad 0], \qquad M = [M_s \quad M_o \quad I], \tag{16}$$

where s is the index of the part in the basis, and o is the index of the part not in the basis. By (14) and (16),

$$M_s\Pi_s' + M_o\Pi_o' = E_i' \tag{17}$$

and

$$\begin{aligned} \Pi_s' &= M_s^{-1}E_i' - M_s^{-1}M_o\Pi_o' \\ &= M_s^{-1}E_i' \end{aligned} \tag{18}$$

since $\Pi_o = 0$.

It was shown by Wolfe and Danzig [9] that the linear programming procedure assures that, in (18), $M_s^{-1} = [(I-\alpha P_s)^{-1}]'$, where P_s is a transition matrix selected from R. This means that there will be exactly one column in M_s for every possible starting state. We repeat, for completeness, the es-

Table 2. First and last simplex sections[a]

	C_s			State 1			State 2			"Slacks"	
		$C\rightarrow$		5.0	4.5	0.0	2.0	2.3	0.0	0	0
		Basis	Π_s	a_1	a_2	a_3	b_1	b_2	b_3	d_1	d_2
First section	0	d_1	1	0.82	1.00	0.10	−0.54	−0.36	0.00	1	0
	0	d_2	0	−0.72	−0.90	0	0.64	0.46	0.10	0	1
Last section	5.0	a_1	4.706	1.00	1.133	0.471	0.00	0.132	0.397	4.706	3.971
	2.0	b_1	5.294	0.00	−0.133	0.529	1.00	0.868	0.603	5.294	6.029
		z_j	34.118	5.0	5.397	3.412	2.0	2.397	3.191	34.118	31.912
		z_j-c_j		0.0	0.897	3.412	0.0	0.097	3.191	34.118	31.912

[a] Based on Table 1, with $\alpha=0.9$. For additional explanations see text.

sence of the proof: since $E_i \geq 0$ and $\Pi_o = 0$, then (14.c) and (17) can be simultaneously maintained only if every row of M_s contains at least one nonnegative element. The only positive elements in M are of the form $1-\alpha p_{ii}{}^d$, of which there is one in every column. The matrix M_s is of the order n; it has n columns, each with exactly one element of the form $1-\alpha p_{ii}{}^d$. It also has n rows, and must, as stated, have at least one nonnegative element in every row. Hence, it will have exactly one element of the form $1-\alpha p_{ii}{}^d$ in every row. Therefore, there will be exactly one element $1-\alpha p_{ii}{}^d$ in every row and column of M_s, which completes the proof.

Equation (18) can now be written as

$$\Pi_s' = [(I - \alpha P_s)^{-1}]' E_i', \tag{19}$$

and, therefore,

$$C\Pi' = C[(I - \alpha P_s)^{-1}]' E_i'. \tag{20}$$

Comparing (20) to (5), we see that $C\Pi'$ is the worth of a Markov process currently in state i. The maximal value of $C\Pi'$—the value of the objective function in the solution to (14)—is the maximal worth of a system of Markov processes.

The solution to (14) determines a policy vector, D_s, which can be constructed by observing the vectors in the basis. It stems from Property 7 of the next section that D_s is not affected by the starting state of the process. Thus, D_s of linear programming, like D^* of the dynamic programming solution, is an optimal policy vector. The same expected maximal present value is reached by the linear and the dynamic programming methods and, if there is only one unique optimal policy vector, then $D_s = D^*$.

In the next section we shall investigate some of the properties and possible interpretations of the simplex routine and elaborate further on the lines of similarity between the dynamic and the linear programming methods.

Properties of the Simplex Solution

It will be convenient if we state here the criterion function of the simplex routine—the $Z-C$ row vector—

$$\begin{aligned} Z - C &= C_s M_s^{-1} [M_s \; M_o \; I] - [C_s \; C_o \; 0] \\ &= C_s [I \; M_s^{-1} M_o \; M_s^{-1}] - [C_s \; C_o \; 0] \\ &= [0 \; C_s M_s^{-1} M_o - C_o \; C_s M_s^{-1}]. \end{aligned} \tag{21}$$

Reference to the element of (21) is made in the discussion that follows.

Property 1

As was previously explained, by programming for a D_s we select a transition matrix P_s and $M_s = (I - \alpha P_s)'$. Therefore, by (7),

$$
\begin{aligned}
(22)\qquad M_s^{-1} &= [(I - \alpha P_s)^{-1}]' \\
&= [(I - Q)^{-1}]' \\
&= V',
\end{aligned}
$$

where V is the fundamental matrix associated with the "transient" matrix αP_s. Thus, in Table 2, consistent with the terminology introduced in the section "Markov Chains in Economic Systems," the expected discounted number of times that a process, currently in state 2, will be in state 1 is 3.971, and in state 2 is 6.029.

Property 2

By equations (22) and (9), the sums of the columns of M_s^{-1} are $1/(1-\alpha)$. In Table 2, $\alpha=0.9$, $1/(1-\alpha)=10$, and the sums are

$$\text{column } d_1\text{:}\quad 4.706 + 5.294 = 10$$

$$\text{column } d_2\text{:}\quad 3.971 + 6.029 = 10.$$

Property 3

Let $u_{ik}{}^o$ be the simplex table element for row i, state k, and o a value for $d(k)$ outside the basis. Thus $u_{ik}{}^o$ is defined by $M_s^{-1}M_o = [u_{ik}{}^o]$. For example, in Table 2, column b_2, last section, $u_{12}{}^2 = 0.132$.

By Property 1, $M_s^{-1}M_o = V'M_o$. Therefore, in scalar notations and denoting by $p_{ij}{}^o$ the transition probabilities in M_o (thus $p_{ij}{}^o$ is the probability of transition from i to j with action o),

$$
\begin{aligned}
(23)\qquad u_{ik}{}^o &= -\sum_{j\neq k} v_{ji}\alpha p_{kj}{}^o + v_{ki}(1 - \alpha p_{kk}{}^o) \\
&= v_{ki} - \alpha \sum_j p_{kj}{}^o v_{ji} \qquad (k = 1, 2, \cdots, n).
\end{aligned}
$$

Examining the last line—the two-steps form of (23)—one recognizes that $u_{ik}{}^o$ is the difference between (*a*) the expected discounted number of times that a process, currently in state k, will be in state i—if the present policy is adopted (v_{ki}), and (*b*) the expected discounted number of times that a process starting in state k will be in state i if the test policy, with action $d(k)=o$ for the current stage and the basic policy for all future stages, is adopted. Action o is taken *once* and the basic policy D_s is followed for all other stages. Hence, $u_{ik}{}^o$ is the marginal rate of substitution of the present (basic) policy to the *alternative policy* with action o for state k in all stages. The substitution is in the decision vector D, and it is "marginal" in that the alternative policy is adopted for only one stage—the current stage.

Property 4

The sum of the elements in every column of the simplex table is unity.

For actions in the basis this is obvious—these columns are unit columns. For actions not in the basis, the sums of the elements of the matrix $M_s^{-1}M_o$ are also unity. Since the sum of every column of the matrix M is $1-\alpha$, therefore, by *A.2* in the Appendix, the sums of the columns of M_s^{-1} are all $1/(1-\alpha)$. Hence, by *A.1* of the Appendix, the column sums in $M_s^{-1}M_o$ are

$$(1-\alpha)/(1-\alpha) = 1.$$

For example, in Table 2, column a_1, the sum is

$$1.133 - 0.133 = 1.0.$$

Making use of (23), we write the column sum as

$$\sum_i u_{ik}^o = \sum_i \left(v_{ki} - \alpha \sum_i p_{kj}^o v_{ji}\right) \tag{24}$$
$$= 1 \qquad (k = 1, 2, \cdots, n).$$

The sum in the right-hand side of the first line of (24) is the difference in the total discounted number of stages under the two policies—the basic policy and the test policy. In general, the total discounted number of stages is the same under any policy (Property 2). The difference in (24), which is unity, stems from the fact that the count of stages for the basic policy includes the current stage (the sum in equation 7, for example, goes from *zero* to infinity), whereas for the test policy the count starts from the next stage and omits the current one.

Property 5

The dual values, the elements of the row vector $C_sM_s^{-1}$, are the values of the alternative objective function, under the basic policy, for all possible starting states. If we write the element k of this vector as z_k^s and denote by c_i^s the element of C_s, the dual values are

$$z_k^s = \sum_i c_i^s v_{ki} \qquad (k = 1, 2, \cdots, n), \tag{25}$$

which is exactly (8′). In the table, $z_1^s = \$34.118$—the value of the objective function for a process starting in state 1; $z_2^s = \$31.912$—the objective function for a process starting in state 2.

Property 6

The elements in the $Z-C$ row for actions not in the basis (21) are $C_sM_s^{-1}M_o - C_o$.

For a state k and action o, we shall denote these elements in the criterion function as $z_k^o - c_k^o$ and write in scalar notation

$$(26) \quad z_k{}^o - c_k{}^o = -\sum_{j \neq k} \alpha p_{kj}{}^o z_j{}^s + z_k{}^s(1 - \alpha p_{kk}{}^o) - c_k{}^o$$

$$= z_k{}^s - \left(c_k{}^o + \alpha \sum_j p_{kj}{}^o z_j{}^s\right) \qquad (k = 1, 2, \cdots, n).$$

The term in the parenthesis in the second version of (26) is the two-steps form of the objective function, for a process in state k, under the test policy. The alternative policy—with action o for state k—will be adopted throughout all future periods if the value of (26) is negative, that is, if the test policy is superior to the basic policy. Since the process lasts forever, if action o for state k is superior for the current state it will also be superior in any future state. This principle is, of course, the rationale behind the dynamic programming procedure, outlined in the section, "Dynamic Programming." It is evident now that the criteria for changing a policy, from iteration to iteration, are the same in the linear and in the dynamic programming techniques. The one difference, however, is that in the simplex method of linear programming one element of D is replaced at a time, whereas in dynamic programming a new vector D is constructed at every iteration, which can differ from the previous policy by several elements.

Property 7

The optimal policy is not affected by the starting state of the process. To see this, one must show that a change of E_i to E_j will not alter the basis of the linear programming solution. Denote a solution vector associated with the starting state i by $\Pi_s(i)$. We know [5, p. 133] that

$$(27) \qquad \Pi_s(i) = M_s{}^{-1}E_i{}' \geq 0$$

is a feasible solution for a starting state i, and that a change of E_i to E_j will not alter the optimal basis, M_s, if, in addition to (27),

$$(28) \qquad \Pi_s(j) = M_s{}^{-1}E_j{}' \geq 0.$$

The condition in (28) is maintained, since all elements in $M_s{}^{-1}$—the v_{ij} elements—are nonnegative.

Extensions and Applications

A multiprocess system

Generally an enterprise will not be a single process but will constitute a system of many processes—fields in a farm, for example, or machines in a factory, or units of an operating army. If we assume that these processes are independent and let e_i be the number of processes, at present in state i, in an enterprise, then the total worth of the enterprise is

$$W = \sum_{i=1}^{n} e_i z_i^s, \tag{29}$$

where z_i^s is defined as in (25). W can be easily calculated from the dual values of the linear programming solution.

Alternatively, a direct approach can be implemented: define a state vector $E(1 \times m)$ whose elements are the e_i values (the vector E_i is now a particular value of E), and instead of (14) solve as follows:

$$(14') \quad \begin{cases} a. \ a \max C\Pi' \\ \quad \text{subject to} \\ b. \ M\Pi' = E' \\ c. \ \Pi \geq 0. \end{cases}$$

The maximal value of the objective function in (14′) will be the W of (29).

A decomposable system

Up to now, we have assumed a system that is not decomposable. This need not be the only case. If the matrix M is decomposable, and if, say, $E_i = E_1$, then (14.b) will be

$$\begin{bmatrix} M_1 & 0 \\ & \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \Pi_1' \\ \\ \Pi_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{30}$$

The elements of Π_2 in (30) must be zeros by the formulation of the problem. The second chain will not be programmed at all.

To avoid this difficulty, it has been suggested [4, 7] that, even in cases of single-process systems, (14′) be solved with an arbitrary nonzero E—the vector on the right-hand side. The optimal policy is not affected by this device. The calculated value of the objective function depends, of course, on the selected values for E.

An inferior state

Another assumption was that chains were regular, that their fundamental matrices had no zero entries, that all states were probable far enough in the future. In practice, one might encounter states which are economically inferior and can be avoided—small inventories, for example, or old machinery. If it is possible, and the appropriate actions are specified, a policy will be selected that will avoid the inferior states. If the process is started in such a state, it will leave that state in one or a few periods. As an example, consider, in Table 3, a new expanded matrix constructed from Table 1 by eliminating, for simplicity, a_3 and b_3 and adding a third state.

Table 3. An expanded R matrix with an inferior state

States	Actions	Probabilities of transition			Immediate rewards
		to state 1	to state 2	to state 3	
State 1	a_1	0.20	0.80	0.00	\$5.00
	a^2	0.00	1.00	0.00	4.50
	a^4	0.00	0.00	1.00	0.00
State 2	b_1	0.60	0.40	0.00	\$2.00
	b_2	0.40	0.60	0.00	2.30
	b^4	0.80	0.00	0.20	1.00
State 3	c_1	1.00	0.00	0.00	\$4.00
	c_2	0.10	0.70	0.20	4.50

Programming,[7] one finds that the optimal policy vector, D_s, of this process consists of a_1, b_1, and c_1 and the corresponding transition matrix is, therefore,

$$P_s = \begin{bmatrix} 0.20 & 0.80 & 0 \\ 0.60 & 0.40 & 0 \\ 1.00 & 0.00 & 0 \end{bmatrix}.$$

An absorbing state

As experience teaches, some policies may lead to irreversible, and sometimes destructive, results. A particular crop rotation will not protect the soil and a heavy rain may cause erosion and destroy all future possibility of cultivating the field. A monopolist may charge high prices that will breed rival firms. These are breakdown cases whose Markov matrices are like T of the section "Markov Chains in Economic Systems." Some reflection, and the example below, will show that "destructive" policies may sometimes be optimal. In fact, whether they will be chosen or rejected depends, all other things being the same, on the discounting rate—the higher the rate of interest, the more probable it is that a "suicidal" policy, which yields high income until destruction, will be adopted.

As an example, consider the expanded R matrix given in Table 4. Note that the reward for the third, absorbing state is zero and that no possible action is attached to this state, which stands for the collapse of the economic system. The optimal policies for this system are listed in Table 5. Also given in Table 5 are the probabilities that a process starting in state 1 will be in any of the states at some specified t period. Once action a_1 is introduced, the process must end in state 3.

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[7] We took $\alpha = 0.9$ in this case too.

Table 4. Expanded matrix with rewards, possible breakdown case

States	Actions	Probabilities of transition			Immediate rewards
		to state 1	to state 2	to state 3	
State 1	a_1 a_2	0.40 0.70	0.55 0.30	0.05 0.00	\$6.00 4.00
State 2	b_1 b_2	0.30 0.40	0.50 0.60	0.10 0.00	\$5.00 3.00
State 3		0.00	0.00	1.00	\$0.00

The right-hand section of the table lists the expected number of times (not discounted) that the process will be in any of the states, under the optimal policies. The numbers in the parentheses are the standard deviations of these numbers [8, Chap. 3]. Thus, in Table 5, in the lower section, under policy a_1b_1, the number of times that a process starting in state 1 will be in state 2 is 7.35 ± 7.49: the standard deviations are quite high in relation to the expected values. Under policy a_2b_2, the process will never reach the absorbing state and will be an infinite number of times in both states 1 and 2.

Depletion and deterioration

The last section dealt with a system with a possible breakdown case. More probable than the sudden "death" or collapse of the economic process is the possibility of depletion or decay of productivity—the deterioration case. A particular crop rotation will gradually impoverish the field; pumping of coastal groundwater damages the quality of that source; a certain maintenance routine results in a gradual reduction of income from an asset. In some respects depletion and deterioration are "historical" phenomena, alien to the Markovian assumption of independence. However, by utilizing the analogy of the deterioration case to the other two cases (in the section "Income Streams"), one may incorporate realistic types of these phenomena into our model.

Assume, for simplicity, a zero rate of interest, namely $\alpha = 1$, and let income, productivity, service, etc. from the economic process deteriorate at a rate $1-\beta (0<\beta<1)$ per period. Expected, not discounted, worth of the income stream is

$$(31) \qquad z_i = \sum_{t=0}^{\infty} E_i(\beta P)^t C' = E_i(I - \beta P)^t C'.$$

More interesting will be the case in which the rate of deterioration is not

Table 5. Characteristics of optimal policies for data in Table 4

Range of interest rate	State	Optimal policy	Transition matrix			Probability of state (E_1p^t)							Expected number of transitions in state (standard deviation)	
						$t=0$	1	2	3	5	10	∞	State 1	State 2
percent														
0–10	1	a_2	0.70	0.30	0.00	1.00	0.70	0.610	0.583	0.572	0.571	0.571	∞	∞
	2	b_2	0.40	0.60	0.00	0.00	0.30	0.390	0.417	0.428	0.429	0.429	∞	∞
	3		0.00	0.00	1.00	0.00	0.00	0.000	0.000	0.000	0.000	0.000	0	0
11–20	1	a_1	0.40	0.55	0.05	1.00	0.40	0.380	0.372	0.357	0.321	0.000	20 (19.5)	27.5 (25.8)
	2	b_2	0.40	0.60	0.00	0.00	0.55	0.550	0.539	0.517	0.465	0.000	20 (19.5)	30.0 (29.5)
	3		0.00	0.00	1.00	0.00	0.05	0.070	0.089	0.126	0.214	1.000	∞	∞
21 and up	1	a_1	0.40	0.55	0.05	1.00	0.40	0.325	0.295	0.248	0.162	0.000	5.35 (3.64)	7.35 (7.49)
	2	b_1	0.30	0.60	0.10	0.00	0.55	0.550	0.509	0.429	0.280	0.000	4.00 (4.77)	8.00 (7.48)
	3		0.00	0.00	1.00	0.00	0.05	0.125	0.196	0.323	0.558	1.000	∞	∞

just one rate for the process but differs from state to state. Now, at the period in which the process occupies state i, its productivity deteriorates at the rate β_i. For example, expected income from the next stage of a process, currently in state i, is $\beta_i \sum_j p_{ij}c_j = \sum_j \beta_i p_{ij} c_j$, or, in matrix notation, $E_i BPC'$, where B is a diagonal matrix with β_i on the diagonal and zeros elsewhere.

Expected value of an everlasting process is, therefore,

$$(32) \qquad z_i = \sum_{t=0}^{\infty} E_i(BP)^t C' = E_i(I - BP)^{-1}C'.$$

It is easily seen now that to allow nonzero rates of interest, one simply incorporates α in (32) to form

$$(33) \qquad z_i = E_i(I - \alpha BP)^{-1}C'.$$

For alternative policies and programming, B is expanded to allow $\beta_i^{(di)}$—deterioration is a function of state and action.

Growth and appreciation

If deterioration is represented by $\beta_i < 1$, growing productivity or appreciation can be represented by $\beta_i > 1$. In fact, (33) applies to cases of appreciation so long as $\alpha\beta_i^d < 1$ for all i and d. If $\alpha\beta_i^d \geq 1$ for some i and d, the existence of the inverse matrix of (33) is not assured; that is, z_i in (33) need not be finite. Programming is, however, still possible by, for example, considering a finite horizon. We shall not pursue this subject further here.

Concluding Remarks

We have tried to show that the Markov chain model may be used in a variety of economic applications. The discussion of the linear programming solution has facilitated, we trust, better understanding of the Markov process and of the rival dynamic programming method. An unsolved problem is that of the incorporation of the regular linear programming limitations and requirements into the present model. The difficulty lies in the fact that the solutions to the Markovian systems are in terms of expected numbers, while the actual magnitudes will change from period to period and may under- or overshoot limitations and requirements, if such exist. We hope to return to this question in the future.

Appendix

A. 1

Let $B = [b_{ij}]$ and $F = [f_{ij}]$ be n-order square matrices with constant column sums: $\sum_i b_{ij} = s$ $(j = 1, 2, \cdots, n)$ and $\sum_i f_{ij} = t$ $(j = 1, 2, \cdots, n)$. If

we let the matrix $G=[g_{ij}]$ be the product matrix of B and $F(G=BF)$, then the column sums of G are all st.
Proof:

$$\begin{aligned}\sum_i g_{ij} &= \sum_i \sum_k b_{ik} f_{kj} \\ &= \sum_k f_{kj} \sum_i b_{ik} \\ &= s \sum_k f_k \\ &= st.\end{aligned}$$

A. 2

If we let $H=[h_{ij}]$ be the inverse matrix of $B(H=B^{-1})$, then $\sum_i h_{ij}=1/s$ $(j=1, 2, \cdots, n)$.
Proof:

$$\begin{aligned} BH &= I \\ \sum_i \sum_j b_{ij} h_{jk} &= 1 \\ \sum_j h_{jk} \sum_i b_{ij} &= 1 \\ s \sum_j h_{jk} &= 1 \\ \sum_j h_{jk} &= 1/s.\end{aligned}$$

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A Programming Model for Optimal Patterns of Investment, Production and Consumption Over Time*

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ABSTRACT

This paper presents a formulation of an applicable linear programming model for a growing economic unit. The model covers all aspects of activity of such a unit: production, investment, consumption and credit operations. Alternative consumption functions are incorporated. Particular care is taken to make the suggested system consistent with economic theory, and it is shown that correct formulation will yield solutions which maintain temporal equilibrium throughout. Diversions from the correct formulation are also analyzed.

This work is an attempt to construct a programming model of optimal resource allocation in an economic unit over time. Particular care will be taken to make the model practically applicable as well as theoretically sound. There is no need to dwell on the importance of the problem of dynamic resource allocation. It is relevant to the understanding of the behaviour of the household, to the budgeting of the firm and to planning of economic development. We hope that this paper will be a contribution toward better solutions of problems in these areas.

Our point of departure will be Hirshleifer's "On the Theory of Optimal Investment Decision" (1958), where a model first suggested by Fisher is extended. This work is well known and it will not be reviewed here. It will suffice to remind the reader that Hirshleifer found that investment decisions should always be made simultaneously with consumption decisions of the economic unit. Hirshleifer's model is a theoretical analysis employing indifference curves and continuous transformation curves. We shall present a translation of his analysis into a mathematical programming model. Such a formulation has already been suggested by Baumol and Quandt (1965), whose model includes the businessman's welfare function in the objective function of the linear programming problem. They made Hirshleifer's case into an applicable programming model only in a very limited sense,** since welfare functions are not observable, and we doubt that many businessmen can formulate their own function.

We shall try to introduce observable consumption functions into the programming model, discuss the difficulties that arise and suggest practical solutions. Our model will be more complete than Baumol and Quandt's in covering financial and current production aspects of the economic unit, as well as investment activities.

The Appendix presents a three-year simplex tableau of what we call Probelm IV which should make it easier for the reader to follow the mathematical formulations of the models.

A PROGRAMMING FORMULATION OF HIRSHLEIFER'S CASE

We start by introducing some notations which will also be used in subsequent sections. We adopt the notational convention of using capital letters for matrices only. Lower case letters are vectors, unless indicated otherwise. Greek letters denote scalars.

Let

x_t^p = level of current production activities at t, $t = 1, 2, \ldots, T$;

x_t^i = level of "real" investment activities started at t;

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** Ophir suggested a capital accumulation model of similar nature, but did not consider consumption as an endogenous variable.

ϕ_{tf} = amount of funds lent ("financial investment") at t;
ϕ_t^b = amount of funds borrowed at t;
y_t = income* at t, $t = 1, 2, \ldots, T-1$;
c_t = consumption at end of year t, $t = 1, 2, \ldots, T-1$;
β_t^f = lending discount rate at t;
β_t^b = borrowing discount rate at t;
ω = wealth at the horizon T;
A_t^p = matrix of input coefficients related to limited factors per unit of current production activities at t (technology matrix);
$A_{\tau t}^i$ = the technology matrix describing input requirements at t per unit of investment activities started at $\tau \leqq t$;
$B_{\tau t}^i$ = a matrix describing outputs of investment activities in the same fashion as $A_{\tau t}^i$ describes inputs;**
k_t^p = pecuniary requirements per unit of current production activities at t;
$k_{\tau t}^i$ = pecuniary requirements at t per unit of investment activities started at $\tau \leqq t$;
μ_t = amount of funds available exogenously at t;
q_1 = available quantities of production factors at $t = 1$;
q_t = the non-obsolete portion of q_1 at t ($t = 2, 3, \ldots, T$);
r_τ^i = residual value at the end of T per unit of investment activities started at $\tau \leqq T$;
Ω = value of q_T at the end of T;
r_t^p = revenue per unit level of current production activities at t.

Thus an investment project, in the present model, does not directly contribute income but adds to the stock of reproducable assets of the economy (see Appendix). It should be noted that we can take care of depreciation by adjusting the elements of q_t and $B_{\tau t}^i$. Anticipated technological changes are indicated by the fact that the technology matrices are subscripted by t.

We do not have to assume constant interest rates. Declining lending or, what is more often the case, rising borrowing rates can be introduced via continuous or step functions without changing qualitatively any of the subsequent results.***

Problem I (Hirshleifer's case) is to maximize

$$F(c_1, c_2, \ldots, c_{T-1}, \omega) \tag{1}$$

subject to

$$-r_{t-1}^p x_{t-1}^p - \beta_{t-1}^f \phi_{t-1}^f + \beta_{t-1}^b \phi_{t-1}^b + c_{t-1} + k_t^p x_t^p + \textstyle\sum_\tau k_{\tau t}^i x_\tau^i + \phi_t^f - \phi_t^b \leqq \mu_t \tag{2}$$

$$A_t^p x_t^p + \textstyle\sum_\tau A_{\tau t}^i x_\tau^i - \sum_\tau B_{\tau t}^i x_\tau^i \leqq q_t \tag{3}$$

$$-r_T^p x_T^p - \beta_T^f \phi_T^f + \beta_T^b \phi_T^b - \textstyle\sum_\tau r_\tau^i x_\tau^i + \omega \leqq \Omega \tag{4}$$

$$t = 1, 2, \ldots, T$$

and non-negativity of all variables, where F is the time preference function.

Some remarks, which will apply to subsequent models as well, are in order. First, for $t = 1$, only the last four terms of (2) are relevant. Secondly, from the formulation it seems as if we assume all credit to be on an annual basis. This is done only for notational convenience. However, even if such an assumption were really necessary it would merely imply a "perfect" finance market in the sense that the borrower is certain of being able to secure whatever amounts he deems profitable in any future year. In general, any short and long run combinations of borrowing or lending options can be introduced as any experienced programmer will recognize.

The present formulation differs from Hirshleifer's in one important aspect. He considers only the transformation curve—the efficiency frontier—and leaves the individual projects that give rise to this curve in the background. This raises difficult questions for longer then two year periods (see Bailey's extension of Hirshleifer's analysis (*Bailey*, 1959)) which are not encountered in this programming model. As usual, the individual activities are independent; that is, projects are not mutually exclusive. Mutual exclusiveness may be treated by formulating alternative programs and choosing the best (compare to *Hirshleifer*, 1958, fig. 5).†

* Both y_t and c_t are, of course, scalars.

** The assumption that inputs and outputs of investment activities can be described by disjoint matrices is made to avoid more notational complications.

*** For example see Yaron and Heady (1961), Plessner and Heady (1965).

† See also Weingartner (1966), who discusses extensively programming of mutually exclusive investment projects, but assumes an exogenously given discount rate.

Finally, even if the time preference function is given, application involves difficult problems. Future prices and technology, and in particular residual values of assets (r_τ^i), have to be assessed. This, however, is common to all long run practical models; and, in fact, every businessman making an important decision is, explicitly or implicitly, predicting future economic magnitudes.

Since the internal rate of return plays a pivotal role in the subsequent discussion, we think it instructive to show how it may be theoretically calculated from the solution to the problem. To this end we associate with (2) the "shadow prices" λ_t^μ, with (3)—the vector v_t, and with (4)—the imputed value η.

Consider the lending activity of year T. We define the internal rate of return, in an obvious way, by

$$1 + \rho_T^f = \beta_T^f .$$

If lending actually takes place, then the dual equation associated with the activity reads

$$\lambda_T^\mu - \beta_T^f \eta = 0$$

from which

$$1 + \rho_T^f = \beta_T^f = \frac{\lambda_T^\mu}{\eta} . \tag{5}$$

Next, consider an investment activity started at the beginning of T. We define

$$1 + \rho_T^i = \frac{r_T^{i\cdot} - (a_{TT}^i v_T/\eta)}{k_{TT}^{i\cdot}} , \tag{6}$$

where $r_T^{i\cdot}$ and $k_{TT}^{i\cdot}$ are elements of r_T^i and k_{TT}^i, respectively, a_{TT}^i being a vector of A_{TT}^i. ($r_T^{i\cdot}$, $k_{TT}^{i\cdot}$ and $a_{TT}^{i\cdot}$ belong, of course, to *the same* activity column. We avoided the identifying column index to simplify notation.)

This definition is, to be sure, a common one. The division of v_T by η is appropriate because the costs of inputs have to be expressed in values of the end of T. While v_T is in terms of dollars at the horizon, η is the value to the economic unit of a dollar at the end of T.

If the activity under consideration is operated, the relevant dual equation reads—

$$k_{TT}^{i\cdot}\lambda_T + a_{TT}^i v_T - r_T^{i\cdot}\eta = 0$$

from which, taken together with (6),

$$1 + \rho_T^i = \frac{\lambda_T^\mu}{\eta} . \tag{7}$$

A comparison of (5) and (7) reveals

$$1 + \rho_T^i = 1 + \rho_T^f .$$

In general, it is easy to check that the internal rate of return will be the same for every investment project which is undertaken and will be greater than or equal to the "going" (market) interest rate in lending opportunities.

An internal interest rate is also implicit in every production activity. We define this rate in year t, ρ_t^p, by

$$1 + \rho_t^p = \frac{r_t^{p\cdot} - a_t^p(v_t/\lambda_{t+1}^\mu)}{k_t^{p\cdot}} . \tag{8}$$

Writing the appropriate dual equation, and assuming that production takes place, one finds

$$\frac{\lambda_t^\mu}{\lambda_{t+1}^\mu} = 1 + \rho_t^p . \tag{9}$$

Similarly to (8), we can define $1 + \rho_t^i$ for every t—the internal rate of returns of investment activities—which will also satisfy (9). The equality of the rates of all operated activities is kept, as the reader may check, for every year t. It is, therefore, possible to define a *common* rate of discount, ρ_t^* for the year t,

$$\rho_t^* = \frac{\lambda_t^\mu}{\lambda_{t+1}^\mu} - 1 . \tag{10}$$

Also,

$$\beta_t^b \geqq 1 + \rho_t^* \geqq \beta_t^f , \tag{11}$$

as Hirshleifer has shown. From (7), (9) and (10) we get

$$\lambda_t^\mu = \eta \prod (1 + \rho_\tau^*) , \tag{12}$$

the financial shadow prices are the compound rates of interest multiplied by the value of the dollar at the horizon.

Finally, an immediate result from the dual equations is that the marginal rates of substitution between consumption in any two successive periods is given by

$$-\frac{\partial F/\partial c_t}{\partial F/\partial c_{t+1}} = \frac{dc_{t+1}}{dc_t} = -\frac{\lambda_{t+1}^\mu}{\lambda_{t+2}^\mu} = -(1 + \rho_{t+1}^*) , \tag{13}$$

as one would expect.

PREDETERMINED CONSUMPTION OUTLAYS

As pointed out already, time preference functions are generally unknown. In the simplest alternative model we suggest, consumption is only a function of time and independent of income.

Let c_t^* denote the minimum annual consumption requirement. Then our Problem II is to find non-negative values which maximize—

$$\psi = \omega \tag{14}$$

subject to (2), (3), (4) and

$$c_t \geqq c_t^* . \tag{15}$$

One important characteristic of Problem II is that here, unlike in Problem I, we have at optimum

$$\eta = 1 . \tag{16}$$

This puts us in a situation of having our programming horizon as a definite zero point on the time axis such that one dollar at that point is worth exactly one dollar. In view of (16), we have

$$\lambda_T^\mu = 1 + \rho_T^*$$

and (9), (10), (11) and (12) can be verified to hold.

The major conceptual difference between the two problems arises from the fact that in the latter consumption is treated as a burden on the system, and is not being solved for endogenously. The burden can be forcefully demonstrated if we associate with (15) the dual value λ_t^c and note that

$$\lambda_t^c = \lambda_{t+1}^\mu . \tag{17}$$

It is interesting to point out, that (17) implies

$$-\frac{\partial\psi/c_t^*}{\partial\psi/\partial c_{t+1}^*} = \frac{dc_{t+1}^*}{dc_t^*} = -\frac{\lambda_t^\mu}{\lambda_{t+1}^\mu} = -(1 + \rho_{t+1}^*) . \tag{18}$$

Obviously, (18) is not the same as (13). Both, however, are necessary (though not sufficient) conditions for inter-temporal rquilibrium in consumption.

OBSERVABLE CONSUMPTION FUNCTIONS

Problem I, with utility function as the objective function, is not applicable (but see *Baunol and Quandt*, 1965). Problem II formulation, on the other hand, is unsatisfactory. Theory and practice teach that consumption is a function of income and wealth and not just a predetermined outlay, We shall now present two versions of our model with consumption as an endogenous variable. In the first version we assume the Keynsian consumption function,

$$c_t = \alpha_t + \gamma y_t . \tag{19}$$

In the second case we let consumption be a function of the worth of the assets of the programmed economy at the horizon,

$$c_t = \alpha_t' + \kappa\omega . \tag{20}$$

Both (19) and (20) are observable. Similar functions have already been estimated (*Ferber*, 1966). They are suggested here as "proxies" for the unobservable welfare function.

We now turn to incorporate these functions in our model. Starting with (19), note that income is, in our case, a linear combination of the operated activities.

$$\begin{aligned} y_t &= (r_t^p - k_t^p)x_t^p + (\beta_t^f - 1)\phi_t^f - (\beta_t^b - 1)\phi_t^b \qquad (21) \\ &\equiv z_t x_t^p + \rho_t^f \phi_t^f - \rho_t^b \phi_t^b . \end{aligned}$$

This makes possible the formulation of Problem III: Maximize (14) subject to (3), (4) and

$$\begin{aligned} &-(r_{t-1}^p - \gamma z_{t-1})x_{t-1}^p - [1 + (1-\gamma)\rho_{t-1}^f]\,\phi_{t-1}^f + \\ &+ [1 + (1-\gamma)\rho_{t-1}^b]\,\phi_{t-1}^b + k_t^p x_t^p + \textstyle\sum k_{t\tau}^i x_\tau^i + \\ &+ \phi_t^f - \phi_t^b \leqq \mu_t - \alpha_{t-1} . \qquad (2') \end{aligned}$$

Consumption appears in Problem III, implicity, as a leakage—in (2′) only the amounts not consumed (gross saving) are carried over to next year. Equation (2′) makes sure that the funds diverted to consumption will be determined by (19).

The formulation of Problem III is quite convenient. It will, however, cause misallocation of resources in the programmed economy. To see this, we turn to the dual.

As in Problem II, $\eta = 1$. Internal rates of return are defined as in (6) and (8). However, writing the dual equation for an operated production activity, we get [λ_t^μ is now the shadow price associated with (2′)]

$$k_t^p \cdot \lambda_t^u + a_t^p v_t - (r_t^p - \gamma z_t)\lambda_{t+1}^\mu = 0 \tag{22}$$

and

$$\frac{\lambda_t^u}{\lambda_{t+1}^u} = \frac{r_t^{p\cdot} - (a_t^p v_t / \lambda_{t+1}^u)}{k_t^{p\cdot}} - \gamma \frac{z_t}{k_t^{p\cdot}} =$$

$$= 1 + \rho_t^p - \gamma \frac{z_t}{k_t^{p\cdot}} \qquad (23)$$

The value of $1 + \rho_t^p$ is the marginal rate of return of a production activity. The ratio $\lambda_t^u / \lambda_{t+1}^u$ is the same for all activities. Since the last term of the right hand side of (23) will vary from one activity to another, it is evident that the marginal rate of return of the different production activities will not be equal at the optimum solution to the present problem. It is also possible to show discrepancies between the internal rates of return and the ratios of the shadow prices for the other kinds of activities. They will, however, not appear in investment activities, which do not contribute to current income and have, therefore, no consumption elements in their columns. The economic reason for these findings can easily be explained. In our model consumption is imposed as an income tax. By investing in real assets the programmed unit can avoid the penalty of the tax. However, we are trying to program an economy which regards consumption as "good" not as "bad." Given that consumption contributes to the welfare of the programed economy, resources will be misallocated by the program.

The direction of this misallocation can also be recognized from (23). z_i is the value added in a production activity in year t. The higher the ratio of value added to initial pecuniary requirements (the ratio $z_i / k_t^{p\cdot}$), the larger this discrepancy of the marginal contribution of the respective activity from the overall marginal rate of return of the program. In other words, the solution to Problem III will favor activities with relatively low value added, thus avoiding consumption.

Some comfort can, however, be derived from the fact that business executives of large corportations may still find the formulation of Problem III useful. Assume that we are programming such an enterprise and that c_t is not consumption, but dividends paid in year t, which the management promised stock holders to pay as a function of income. If management does not attach any value to these dividends, it will try to follow the recommendations of a model such as our Problem III.

ELIMINATING MISALLOCATION

Problem III can be amended to eliminate its misallocative nature. Let Problem IV be: Maximize

$$\psi = \sum_t \delta_t c_t + \omega \qquad (24)$$

subject to (2′), (3), (4) and

$$-\gamma z_t x_t^p - \gamma \rho_t^f \phi_t^f + \gamma \rho_t^b \phi_t^b + c_t \leqq \alpha_t . \qquad (25)$$

Eq. (25) is the consumption function incorporated in the linear programming formulation. The δ_t are arbitrary positive constants.

There will be no misallocation of resources if the objective function coefficients in (25) are chosen in such a way that in the solution to Problem IV

$$\delta_t = \lambda_{t+1}^u \text{for every } t. \qquad (26)$$

We shall call a solution maintaining (26) *an unbiased solution.* It is our assumption that in the system we program an unbiased solution exists. The assumption is based on the expectation that in the real world an unibased program exists. We shall later show that such a solution can be found.

To see that misallocation has been eliminated, consider the dual equations corresponding to the c_t columns of Problem IV. They are

$$\lambda_t^c = \delta_t . \qquad (27)$$

Equality will always be maintained, since $c_t > 0$ for every t. (This point is seen clearly from the simplex tableau in the Appendix. The activities c_t do not require factors of production, but contribute to the objective function.)

The dual equations of the production activities will now contain an element from (25). Instead of (22) we have

$$k_t^{p\cdot} \lambda_t^u + a_t^p v_t - (r_t^{p\cdot} - 8 z_i) \lambda_{t+1}^u + \gamma z_i \lambda_t^c = 0. \quad (22')$$

However, if $\delta_t = \lambda_{t+1}^u$ one gets $\lambda_t^c = \lambda_{t+1}^u$. Introducing this last equality into (22′) will eliminate altogether the misallocation term $\gamma(z_i / k_t^{p\cdot})$ that appeared in (23).

The economic interpretation is simple. In an unbiased solution consumption dollars are given the same weight in the objective function as dollars carried over to next year's production and investment activities. Thus consumption is no longer a burden. The equality of the transformation rates in production and consumption can be shown here, as in (18). Also, the remark we made at the end of Section B, about the maintenance of the condition for equilibrium, can be repeated more emphatically with respect to Problem IV.

We now show that an unbiased solution (whose existence we assumed) is attainable. Our procedure will simply be to program with alternative δ_t values. We shall presently claim that if we try enough, we may find the set of δ_t values that will maintain the equality of (26). We do not suggest a practical or efficient search method, we just want to show that it is possible to reach an unbiased solution.

Proposition. An unbiased solution consistent with (26) is attainable in a finite number of computational operations.*

To prove the proposition, we start by showing that the λ_t^μ values are restricted to a bounded and *identifiable* region. This is done by inserting (10) into (11) (both equations apply in Problem IV too)

$$\beta_t^b \geqq \lambda_t^\mu/\lambda_{t+1} \geqq \beta_t^f. \tag{28}$$

For $t = T$ (recall, $\eta = 1$)

$$\beta_T^b \geqq \lambda_T^\mu \geqq \beta_T^f. \tag{29}$$

Since the λ_t^μ are compounded interest rates (12), boundedness is not surprising—the annual internal rates are bounded between the lending and the borrowing rate. Reference to financial operations was made for convenience. Zero will always be a lower bound and an upper bound can be found by considering the production or investment activity with the highest returns to the dollar when all shadow prices of production factors are zero.

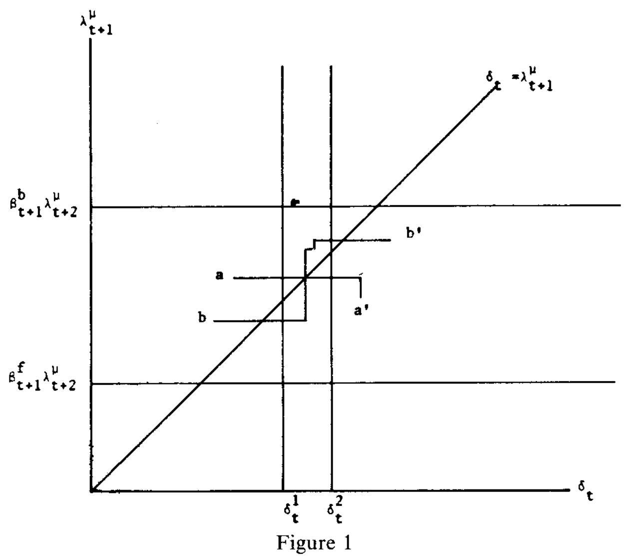

Figure 1
An Unbiased Solution

It stems from the nature of linear programming that there exists a set $\{\varepsilon_t : \varepsilon_t > 0\}$ such that the two sets of objective function coefficients $\{\delta_t\}$ and $\{\delta_t + \varepsilon_t\}$ will yield identical solutions to Problem IV. (Identical solutions are here, of course, identical in the values of the primal solution.) This feature assures that an unbiased solution can be found.**

Suppose that we partition arbitrarily the range that δ_t can take and solve all alternative linear programming problems with the different δ_t values. Suppose also that two such alternative programs yielded identical primal solutions. In Figure 1, let δ_t^1 and δ_t^2 be the alternative objective function coefficients, for a year t, associated with *the same* primal solution. The solution of the dual problem will now indicate whether or not an unbiased solution has been reached.

The graphs aa′ and bb′ represent the possible forms of variation of λ_{t+1}^μ—the coefficient of the dual solution—occasioned by variations in δ_t. In the case depicted by bb′ $\delta_t - \lambda_{t+1}^\mu$ changes sign as we go from δ_t^1 to δ_t^2 which, as seen in Figure 1, is the criterion for an unbiased solution. The reasoning for the case of aa′ is even simpler.

It is now clear that a *finite* partition of the range that the values of the δ_t's can take, can be found, such that if all combinations of these values are tried in objective functions of alternative programs, an unbiased solution will be attained. This proves the proposition.

The unbiased solution need not be unique. Perhaps in most cases the programmer will choose the solution with the highest ω—wealth at the horizon.

CONSUMPTION AS A FUNCTION OF WEALTH AT THE HORIZON

We may leave it to the interested reader to detail the formulation of Problem V—a programming model with (20) as the consumption function. We will present

* A solution consistent with (26) is any solution identical with the unbiased solution. The meaning of this will become clear shortly.

** Some difficulties may be encountered in cases of degenerate solutions. The Proposition will still hold. The proof is, however, rather tedious and will not be given here.

here only a series of remarks. (a) No misallocation problem arises since consumption is affected by final wealth only. (b) Equality of rates of transformation in consumption and production is maintained here too. (c) Wealth is taken here in the Friedman (1957) sense, and includes reproduceable as well as non-reproduceable assets of the economy. (d) No special search procedures are required in this case. However, this advantage is gained at the expense of difficulties associated with the evaluation of wealth.

CONCLUDING REMARKS

We have tried to show in this paper that dynamic capital programming models, consistent with economic theory, can practically be applied. For business analysis Problem II, with predetermined consumption outlays, can be a useful framework. Several consumption (or any other "leakage") patterns may be tried, and management will make the choice among the alternative programs.

The incorporation of consumption as an endogenous variable can be of crucial importance in the context of development. It should, however, be noted that the models discussed thus far are appropriate to the analysis of regional development problems. When it comes to national development, it is probably inappropriate to disregard demand functions for the commodities produced, as we did.

APPENDIX

SIMPLEX TABLEAU, PROBLEM IV.

Objective Function	*Dual*	0	0	0	0	δ_1	0	0	0	0	δ_2	0	0	0	0	1	*Right Hand Side*
	Varia.	x_1^p	x_1^i	ϕ_1^f	ϕ_1^b	c_1	x_2^p	x_2^i	ϕ_2^f	ϕ_2^b	c_2	x_3^p	x_3^i	ϕ_3^f	ϕ_3^b	ω	*Vector*
1. Finance year 1	λ_1^μ	k_1^p	k_1^i	1	-1												μ_1
2. Real input year 1	v_1	a_1^p	a_1^i														q_1
3. Consumption year 1	λ_1^c	$-\gamma z_1$		$-\gamma\rho_1^f$	$\gamma\rho_1^b$	1											α_1
4. Finance year 2	λ_2^μ	$-h_1^p$		$-h_1^f$	h_1^b		k_2^p	k_2^i	1	-1							$\mu_2-\alpha_1$
5. Real input year 2	v_2		$-b_{12}^i$				a_2^p	a_2^i									q_2
6. Consumption year 2	λ_2^c						$-\gamma z_2$		$-\gamma\rho_2^f$	$\gamma\rho_2^b$	1						α_2
7. Finance year 3	λ_3^μ						$-h_2^p$		$-h_2^f$	h_2^b		k_3^p	k_3^i	1	-1		$\mu_3-\alpha_2$
8. Real input year 3	v_3		$-b_{13}^i$					$-b_{23}^i$				a_3^p	a_3^i				q_3
9. Residual value	η		$-r_1^i$					$-r_2^i$				$-r_3^p$	$-r_3^i$	$-\beta_3^f$	β_3^b	1	Ω

NOTES TO THE TABLE

1. The tableau assumes a simple example:
   a. The programming period is three years;
   b. In every year there is one activity of every kind—current production, real investment, lending and borrowing.
   c. There is one real factor which is in limited supply in the first year and is reproducable by the real investment activity.
2. Unlike in the text, all lower case and Greek letters are scalars.
3. $h_t^p = r_t^p - \gamma z_t$

   $h_t^f = 1 + (1 - \gamma)\rho_t^f$

   $h_t^b = 1 + (1 - \gamma)\rho_t^b$

These are gross saving elements [see (2')].

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Paper 5-3

AN ECONOMIC ANALYSIS OF DRAINAGE PROJECTS IN SINKING SOILS*

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INTRODUCTION

An area of approximately 40,000 dunams (10,000 acres) of swamps and a lake in the northern part of the Jordan Basin in upper Galilee was reclaimed in the mid 1950's when the first stage of the Hula Drainage Project was completed. The area has since been under cultivation. However, substantial parts of it suffer from winter floods and additional drainage projects are now being considered. The new project, now under planning and economic evaluation, is a complex system composed of several multi-stage subprojects. This paper develops the framework for the economic analysis of one of these subprojects, namely, the drainage of the peat soils area.

Peat soils form approximately one half of the drained area. These soils are very rich in organic materials—in some cases over 90% by volume—and cultivation created conditions favorable to their decomposition. This results in a gradual sinking of the soils which progresses faster in some parts of the valley than in others due to local conditions. The average rate is estimated to be in the order of 10 cm per annum. This loss of topographic elevation leads to an increase in the area which is lower than the winter

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level of water in the main drainage canals and is consequently subject to flooding. The lower the area, the higher the probability of winter floods and the damage to crops. It is expected that the sinking process will continue for several decades, lowering the area by several meters.

The sinking process can be controlled, to some extent, by special agricultural methods but these are considered expensive in terms of foregone income, and will probably not be used. On the other hand, the drainage canals, cutting through the area, can be deepened to prevent water from overflowing during the winter. This is the essence of the flood control projects now under consideration. Without going into technical details, we make the simplifying assumption that the larger the investment the deeper the canals and the smaller the flood damages.

The peat areas have been surveyed and maps prepared showing the available information on the composition of soil material. The sinking process can thus be forecast. We shall be able to estimate future floods with the existing drainage system or any new one.

The economic problem that emerges is that of determining optimum size and timing of the drainage project. Since the sinking process is gradual and large projects have to be built in stages—for technical and financial reasons—we shall discuss not only the optimum size and timing of a single project but also projects whose rate of construction is adapted to the rate of sinking of the peat soils. Therefore, the model developed is an investment process whose puvpose is to mitigate worsening economic conditions. One can take as additional examples the rate of construction of highways as a function of everincreasing congestion costs, or investment in advertising to remind the market of the existence of products which it otherwise slowly forgets [5].

As the foregoing discussion indicates, the investment projects are regarded as preventive measures and their contribution to the economy is a rising function of the damage or loss they prevent. This connects our analysis to Marglin's [3], who considered investment projects when demand for their product is rising. At this stage, our analysis is, like his, deterministic; which implies, for example, that we use expected values of the flood damages, instead of their distributions, or assume complete knowledge of the investment projects and their effects. It will become clear below that to some extent we also follow the model of capital accumulation developed by Eisner and Strotz [2]. The theoretical part of the article is general and applies to any case of capital accumulation with rising marginal product of capital. We prefer, however, to keep the discussion specific and to restrict it to the case of our particular flood control project. Generalization should follow easily.

The following section presents notation and our assumptions. Section 2 analyzes a single-stage drainage project, Section 3 deals with the multi-stage possibility. A continuous investment process is introduced in Section 4 and an application in Section 5.

1 Notations and Assumptions

Derivatives are indicated by primes, time derivatives by dots.

t calendar time;

r rate of interest.

The state of the area is characterized by the following variables (see Fig. 1):

A maximum potential income from the area (in dollars). In the present study, this is assumed to be independent of time. The assumption of a rising potential income can easily be incorporated [3].

$g(t)$ deterioration of income due to sinking. Since deterioration is a continuing process, we assume $\dot{g} \geqq 0$. Decomposition reduces the peat soil area, uncovering mineral soil. The area that sinks is thus diminishing. We assume, therefore, that $\ddot{g} \leqq 0$.

$A - g(t)$ actual income if no flood-control measures are taken. This value can become negative but then, unless drainage is improved, the area should probably be abandoned.

The project is constructed gradually, investment adding to its size. The flow of investment is, therefore, a measure of the rate of construction. The size of the project is measured in terms of accumulated investment. This creates a difficulty since the cost of construction will usually depend on the rate of investment. We shall distinguish between net and gross cost ([2], p. 471). Only the first is added to the project and can serve as a measure of its growth. This is the amount of "bricks" laid in the project, measured in money terms. The gross cost depends in addition on the rate of construction. This cost is the cost of laying the "bricks", including the value of the "bricks" themselves. It should be emphasized that the distinction drawn is artificial although the problem is real—very slow or very fast construction will generally be more expensive than investment at some optimum pace.*

* The optimum rate of construction depends on two components: (a) The sinking rate—the demand component, and (b) the cost of investment as a function of the rate of construction—the supply aspect.

Thus let

$\dot{w}(t)$ be the rate of construction (net cost) measured in dollars per unit of time (a year, say).
Without loss of generality we assume that we start from a zero size project, so that the size of the project at time t is

$$w(t) = \int_0^t \dot{w}(t)\, dt.$$

$\phi(\dot{w})$ is the gross cost of construction. As explained above, we assume that $\phi(\dot{w}) \geqq \dot{w}$.

The income of the area is a function of time and of the size of the project:

$P(t, w)$ income in dollars (per year).

In this work we assume (as did Marglin [3]) that the effect of the project can be expressed by a function $h(w)$, such that income is separable in the form

$$P(t, w) = A - g(t)\, h(w),$$

where

$h(w)$ is the effective flood control capacity of a project of size w and it is assumed that

$$0 \leqq h(w) \leqq 1, \quad h(0) = 1, \quad h'(w) < 0, \quad h''(w) > 0.$$

The assumptions on the signs of the first and second order derivatives of $h(w)$ are the usual production function assumptions. Engineers agree with these too, although in practice one may encounter regions of decreasing costs and it is not always easy to arrange subprojects in stages so that $h''(w) > 0$. The effect of the function $h(w)$ is illustrated graphically, for a special case, in Fig. 1.

We assume in the following that a flood-control project, once constructed, will last foreover. As service life of projects of this kind, if properly maintained, is very long, this seems a reasonable assumption. Maintenance costs are usually taken by engineers as a fixed percentage of investment outlays and as such they may be included in the construction costs and need not be treated separately.

Special notation is adopted for the discrete multi-stage cases:

t_i date of construction of stage $i (i = 1, 2, \ldots, n)$, $t_0 = 0$,

w_i size of project after the construction of stage i,

x_i investment of stage i, so that $w_i = \sum_{k=1}^{i} x_k$.

For simplicity, we assume a gross cost function of the form $x_i + c$ (where c is fixed cost per stage) for discrete cases.

2 A Single-Stage Project

Valuable insight is gained by starting the discussion with a single stage case. A single stage project of size w_1 will be constructed at time t_1. Present value of net income from the area is given by

$$y = \int_0^{t_1} [A - g(t)] e^{-rt} dt + \int_{t_1}^{\infty} [A - g(t) h(w_1)] e^{-rt} dt - (x_1 + c) e^{-rt_1} \quad (1)$$

Note that $w_1 = x_1$.

y in (1) is to be maximized with respect to t_1 and to w_1. Since A is the maximum annual income, y is bounded for positive r. The necessary conditions for optimum timing and size are $\partial y/\partial t_1 = \partial y/\partial w_1 = 0$. Second order conditions can be shown to hold.

$$\frac{\partial y}{\partial t_1} = 0 \rightarrow r(x_1 + c) = g(t_1) [1 - h(w_1)]. \quad (2)$$

That is, investment will take place when the (annual) interest cost will be equal to the (annual) value of the damage prevented.

$$\frac{\partial y}{\partial w_1} = 0 \rightarrow \int_{t_1}^{\infty} g(t) h'(w_1) e^{-rt} dt = -e^{-rt_1}. \quad (3)$$

The integrand in (3) is the annual value of the damage prevented by the marginal dollar. The integral is thus the marginal value of the investment. It equals, at the optimum, \$ 1 discounted from t_1.

There still remains the question whether to build or not and for this purpose it will be useful to define:

$$D \equiv \int_0^{t_1} [A - g(t)]\, e^{-rt}\, dt;$$

$$E \equiv \int_{t_1}^{\infty} [A - g(t)]\, e^{-rt}\, dt;$$

$$F \equiv \int_{t_1}^{\infty} [A - g(t)\, h(w_1)]\, e^{-rt}\, dt;$$

$$G \equiv (x_1 + c)\, e^{-rt_1}.$$

The economic rent of the project R is the value of the damage prevented.

$$R = F - E - G = \int_{t_1}^{\infty} \{g(t)\,[1 - h(w_1)] - r(x_1 + c)\}\, e^{-rt}\, dt. \qquad (4)$$

Two cases can be distinguished. In one of them—perhaps the flooding of residential areas—the project should be constructed whenever the rent, R, is positive. This will happen if in the solution of (2) and (3) $0 < t_1 < \infty$, since by (2) the integrand in (4) is zero for $t = t_1$ and non-negative for $t > t_1$, since $\dot{g} \geqq 0$. However, in our case there exists the alternative of abandoning the area. Here the criterion for construction should be $F - G > 0$ (note that E may be negative). If $D < 0$, the area will not be cultivated until the completion of the project at t_1.

Construction may have to start immediately (perhaps for political reasons); optimum size is then determined by solving (3) for $t_1 = 0$. Similarly if the solution of (2) and (3) yields $t_1 \leqq 0$ (in this case $g(t)$ should be defined for negative values of t), construction should be immediate and of the same size as if $t_1 = 0$ was forced.

3 Multi-Stage Projects

If division is possible, construction in stages may increase the efficiency of the system. Net income from an n-stage project is

$$y = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [A - g(t)\, h(w_i)]\, e^{-rt}\, dt + \int_{t_n}^{\infty} [A - g(t)\, h(w_n)]\, e^{-rt}\, dt$$

$$- \sum_{n=1}^{n} (x_i + c)\, e^{-rt_i}. \qquad (5)$$

For convenience we define here $w_0 = 0$.

Remember, $w_i - w_{i-1} = x_i$ $(i = 1, ..., n)$, $h(0) = 1$, $t_0 = 0$.

Again,

$$\frac{\partial y}{\partial t_i} = 0 \rightarrow g(t_i)\,[h(w_{i-1}) - h(w_i)] = r(x_i + c) \qquad (i = 1, 2, ..., n). \tag{6}$$

(Since $t_0 = 0$, this variable cannot be included among the parameters of maximization.)

$$\frac{\partial y}{\partial w_i} = 0 \rightarrow \int_{t_i}^{t_{i+1}} g(t)\,h'(w_i)\,e^{-rt}\,dt = e^{-rt_{i+1}} - e^{-rt_i} \qquad (i = 1, 2, ..., n-1). \tag{7a}$$

$$\frac{\partial y}{\partial w_n} = 0 \rightarrow \int_{t_n}^{\infty} g(t)\,h'(w_n)\,e^{-rt}\,dt = -e^{-rt_n}. \tag{7b}$$

The system (6) and (7) is a set of simultaneous equations. In practice one may encounter cases which will make their "step-wise" solution possible. Some examples will illustrate this point.

a) Assume that the size of the stages is predetermined (this will be the situation in the application illustrated below). Then optimum timing is determined by (6), starting from t_1. Equation (6) may be written in the more general form

$$P(t_i, w_i) - P(t_i, w_{i-1}) = r(x_i + c), \tag{6'}$$

which emphasizes that a stage will be added to the project when the additional income due to the prevention of damage is equal to the interest cost of the capital invested at this stage.

b) In another case, the sequence $\{t_i\}$ may be predetermined, perhaps in the form $t_i = i$, or by any other pattern. Then the set (6) is void and (7) can be solved equation after equation, from w_1 to w_n.

Equations (6) and (7) show that the optimum size of the project at point t_i depends, in general, on the planning horizon. It is instructive to note that when either $\{w_i\}$ or $\{t_i\}$ is predetermined, the optimum size or timing of investment is independent of the planning horizon.*

* This conclusion holds only for linear cost functions and not for the general function $\varphi(w)$.

An intuitive explanation for this is connected to the fact that a project of size w_i contributes by preventing damage during period (t_i, t_{i+1}) and also "delivers" a project of size w_i at t_{i+1}.

c) Another interesting case might be the one in which only the date of completion of the project, t_n, is predetermined. Then one can go "backwards" from t_n, first determining w_n then w_{n-1}, t_{n-1}, etc. This method of solution as well as the previous ones can be interpreted as a dynamic programming algorithm [1]. The recurrence relation for the present case (c) is

$$f(t_i, w_i) = \max_{t_i, w_i} \{ \int_{t_i}^{t_{i+1}} [A - g(t)\, h(w_i)]\, e^{-rt}\, dt - (x_i + c)\, e^{-rt_i} + f(t_{i+1}, w_{i+1})\}, \quad (8)$$

where $f(t_i, w_i)$ is the maximum present value of income if the multi-stage project starts at t_i, and is constructed in $n - i$ stages.

Dynamic programming can be applied to the numerical solution of the system (6) and (7) even if these equations must be solved simultaneously and not step-wise in the sense of points (a)–(c) above.

If the date of the final stage is predetermined, the number of stages is dictated by the solution. If, on the other hand, n is given exogeneously, one could search for the corresponding t_n.

Consider the simplest of the multi-stage projects—the two-stage case. The single-stage project of Section 2 can be obtained as the limit of the two-stage project as $t_2 \to \infty$. Thus, if the solution to the maximization of income from the two-stage project yields $t_2 < \infty$, income from this project will be larger than income from the single-stage case. This can be generalized to the multi-stage case.

We may consider a multi-stage process with an infinite number of stages. Then (6) and (7), expressing the necessary conditions for optimal investment, will form infinite sets of equations.

4 Continuous Construction

Within the context of flood control projects, a continuous construction model is perhaps only of theoretical interest. However, it will be an approximate description of a multi-stage discrete model with small intervals between the stages. The solution of the continuous investment case is concise and one may wish to calculate it to gain more insight into the solution of discrete models. In other cases (consider advertising) it may be a closer description of reality then the discrete model.

In the continuous case, we do not speak of fixed costs, c, as in the discrete case, but permit outlays associated with construction to be larger than net investment and depend on the rate of investment. Thus $\phi(\dot{w}) \geqq \dot{w}$. We start, however, with the case $\phi(\dot{w}) = \dot{w}$ and mention the more general, and complicated, case later.

Present value of net income, if $\phi(\dot{w}) = \dot{w}$, is

$$y = \int_0^\infty [A - g(t)\,h(w) - \dot{w}(t)]\,e^{-rt}\,dt. \tag{9}$$

Maximizing y in (9), we use the calculus of variation ([1], p. 40). Let H stand for the integrand in (9), then by the Euler–Lagrange equation

$$\frac{\partial H}{\partial w} - \frac{d}{dt}\frac{\partial H}{\partial \dot{w}} = 0,$$

we obtain

$$h'(w) = \frac{-r}{g(t)}. \tag{10}$$

The end point condition reduces in this case to

$$\lim_{t\to\infty} e^{-rt} = 0,$$

which is automatically satisfied.

From (10)—since $h'(w)$ is a monotonic function—one can deduce the rate of investment $\dot{w}(t)$, once the explicit forms of the functions $g(t)$ and $h(w)$ are given. Equation (10) thus indicates the optimum path of the project's future history.

Some further observations are noted below:

a) Condition (10) can also be obtained from (6)—the first order condition for optimum timing in the discrete case—which can be rewritten as (remember that $c = 0$)

$$g(t_i)\frac{h(w_{i-1}) - h(w_i)}{w_{i-1} - w_i} = -r. \tag{11}$$

Taking the limit of (11) as $w_{i-1} \to w_i$, we get (10). For a similar approach in the context of dynamic programming see ([4], p. 231).

b) The optimum initial size of the project, w_0 at $t_0 = 0$, is given by (10) and it is such that

$$h'(w_0) = -\frac{r}{g(t_0)}.$$

Thus, the process will start with an initial investment of w_0 and then continue in the path dictated by (10).*†

c) It is important to remember that we found in this and other sections the conditions for maximum net income or minimum losses. Denoting by y^* the value of the integral in (9) when investment follows the optimum path dictated by (10), the project will be economically justified only if $y^* - w_0 \geqq 0$.

Note also that the element of the construction cost in (9) is

$$\int_0^\infty \dot{w} e^{-rt}\, dt = -w_0 + \int_0^\infty (rw)\, e^{-rt}\, dt. \tag{12}$$

The right hand side of (12), obtained by integration by parts, is the difference between the service cost of capital invested in the project and the initial investment, w_0.

d) Differentiating (10) with respect to time one gets

$$\dot{w} = \frac{r\dot{g}}{[g(t)]^2\, h''(w)}. \tag{13}$$

By assumption $\dot{g} \geqq 0$, $h''(w) > 0$. So long as $h''(w) < \infty$ and $\dot{g} > 0$ we have $\dot{w} > 0$. That is, construction will proceed continuously. However, it will stop when $\dot{g} = 0$.

The result, stating that $\dot{w} \geqq 0$, is welcome, since the project cannot be scrapped at a price, disinvestment—that is $\dot{w} < 0$, is meaningless.

e) In general, income from the area will not be constant. Differentiating $A - g(t)\, h(w)$ with respect to time, assuming (10), we obtain the rate of change of income along the optimum path

$$\frac{d[A - g(t)\, h(w)]}{dt} = r\dot{w} - \dot{g}h(w). \tag{14}$$

It is not clear what the sign of (14) is.

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* Note that initial adjustment is here instantaneous. This is due to the assumption of $\phi(\dot{w}) = \dot{w}$ (compare with Eisner and Strotz [2]).

† Remember that we do not assume that a project of any size exists beforehand. This point can easily be modified.

f) Part of the foregoing discussion indicates that this is a somewhat degenerate case. Due to the linearity of the cost function in (9), the derivative $\dot{w}$ does not appear in (10), and there is only one optimum path of investment (see also point (*b*) in Section 3).

In the more general case, where $\phi(\dot{w})$ is not a linear function of $\dot{w}$, the present value of the income is

$$y = \int_0^\infty [A - g(t)\, h(w) - \phi(\dot{w})]\, e^{-rt}\, dt. \tag{15}$$

The necessary condition for optimum path is

$$-\phi''(\dot{w})\, \ddot{w} + r\phi'(\dot{w}) + g(t)\, h'(w) = 0, \tag{16}$$

with the end condition

$$\lim_{t\to\infty} \phi'(\dot{w})\, e^{-rt} = 0. \tag{17}$$

Further investigation of Eq. (16) has been deferred to a later work.

5 An Example

The example presented in this section is based on preliminary data from the Hula project and on some arbitrary assumptions. The analysis should not be taken as a recommendation of any sort.

The planned flood control project is divided into five stages (see Table 1). The first stage, if constructed, will reduce the expected flooded area in 1969 from 4,999 dn to 1,589 dn. Cost of construction is* IL 1,540,000 or IL 452 per dunam. Stage 2, if carried out in 1969, will reduce the expected flooded area by 883 dn in that year, at a cost of IL 2,264 per dunam. The marginal cost increases from stage to stage. This is consistent with our assumption of $h''(w) > 0$.

We assume a rate of interest of 10% (8% capital cost and 2% maintenance). At this rate, the present value of a dunam of land "saved" from the floods (in terms of expected value) is IL 1,282. Thus, according to the last column of Table 1, only stage 1 should be constructed in 1969.

Information similar to that given in Table 1 was projected for the period 1969–2000 from technical data. Thus we could estimate future values of the

* IL 3.5 = $ 1; 1 dn = 0.25 acres.

TABLE 1 PROJECT DESCRIPTION (1969)

Stage	Stage Identification in Hula Project	Cost of Project (w_i) (IL '000)	Cost of Stage (x_i) (IL '000)	Expected Value of Flooded Area (dn)	Change in Area Flooded (dn)	Marginal Cost (IL/dn)
0	Present state			4,999		
1	59.15	1,540	1,540	1,589	3,410	452
2	58.65	3,539	1,999	706	883	2,264
3	58.65 — 0.5	4,963	1,424	570	136	10,471
4	58.65 — 0.5+	7,888	2,925	363	207	14,130
5	58.65 — 0.5++	11,110	3,222	303	60	53,700

Notes:
Costs are based on 1969 data;
Project's effect, in terms of area flooded, is for 1969;
1 dunam = 0.25 acres;
Fixed costs $c = 0$.
$ 1 = IL 3.50.

functions $g(t)$ and $g(t)\,h(w)$. At this point, Eq. (6) was utilized to calculate optimum t_i values. This analysis is carried out in Table 2. Potential income from the project area is IL 6,691,000 per annum. If the project is not carried out, the damage in 1969 will be IL 640,000. Construction of stage 1 in 1969 will contribute IL 437,000 of damage prevention at an interest cost of IL 154,000. It should therefore be constructed immediately.

Stage 2 is to be constructed in 1977. This is the first year in which the annual value of the damage prevented by stage 2 will be higher than the interest cost on the investment at this stage. Stage 3 will be constructed in 1987. The calculations were followed up to the year 2000, showing that stage 4 will not be constructed in this period. The resulting income flows were plotted in Fig. 1.

6 Concluding Remarks

This paper has presented a theoretical framework for the analysis of flood-control projects in the Hula peat soils, and, we trust, for some other cases as well. It serves as a starting point for further research and as a guide to the empirical work which is now in progress.

TABLE 2: CONSTRUCTION PROCESS (IN THOUSANDS OF ISRAELI POUNDS)

Year (t)	1969	1977	1987	2000
Present state—no construction				
Net income $[A - g(t)]$	6,051	5,557	4,852	4,129
Damage $[g(t)]$	640	1,134	1,839	2,562
Stage 1				
Net income $[A - g(t)\,h(w_1)]$	6,488	6,301		
Damage $[g(t)\,h(w_1)]$	203	390		
Damage prevented if stage constructed	437			
Interest cost (rx_1)	154			
Stage 2				
Net income $[A - g(t)\,h(w_2)]$		6,500	5,968	
Damage $[g(t)\,h(w_2)]$		191	723	
Damage prevented if stage constructed		199		
Interest cost (rx_2)		200		
Stage 3				
Net income $[A - g(t)\,h(w_3)]$			6,109	5,075
Damage $[g(t)\,h(w_3)]$			582	1,616
Damage prevented if stage constructed			141	
Interest cost (rx_3)			142	
Stage 4				
Net income $[A - g(t)\,h(w_4)]$				5,079
Damage $[g(t)\,h(w_4)]$				1,612
Damage prevented if stage constructed				4
Interest cost (rx_4)				293
Optimum size of project (w_i)	1,540	3,539	4,963	4,963

Notes:
Potential income: $A =$ IL 6,691,000;
Damage prevented: $P(t_i, w_i) - P(t_i, w_{i-1}) = g(t_i)\,[h(w_{i-1}) - h(w_i)]$;
Column headings show construction dates, except for 2000;
A rate of interest $r = 0.10$ is assumed.

Further work in this study will be in three directions: a) The integration of the analysis of the peat soils project with the analysis of the rest of the Hula Basin drainage system; b) The incorporation of elements of uncertainty and accumulated information in the analysis; c) Extension of the analysis of Section 4 to a more general cost function.

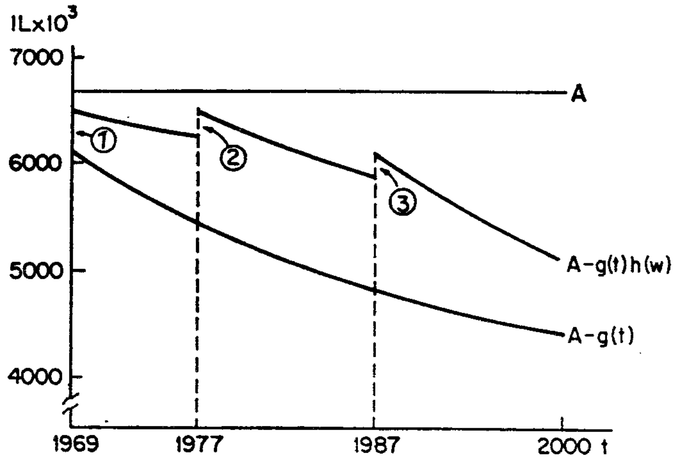

Fig. 1. Future income flows in project area

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AN APPLICABLE LINEAR PROGRAMMING MODEL OF INTER-TEMPORAL EQUILIBRIUM

Yoav Kislev and Yakir Plessner (*)

The problem of consumption over time in a world of certainty in which funds can be borrowed and lent, has been treated in a general fashion by Hirshleifer [6], whose analysis is an extension of the Fisherian theory. Basically, Hirshleifer's model consists of the consumer's time preference function in its most general form and a return-on-investment function. Of these only the second, of course, lends itself to direct application.

Baumol and Quandt (BQ) [1] presented a linear programming reformulation of Hirshleifer's model. While very constructive, their approach is not practical since it assumes knowledge of the utility function (and that it is linear). Further sources of illumination are provided by the studies of Charnes, Cooper and Miller (CCM) [3] and Ophir [10] who, ignoring consumption and using a profit function as their objective in place of the utility function, demonstrated the richness of information obtainable from a linear programming approach.

Two types of attempts to introduce consumption into multi-period linear programming, without impairng applicability i.e., without resorting to unmeasurable utility functions, should be mentioned. The first, used by Loftsgard and Heady [8], was to impose predetermined consumption outlays which are thus independent of income, interest rates, etc. Basically, this is the CCM model with a neutral « tax » on the program. The second approach reintroduces consumption via the Keynesian, linear, consumption function. While feasible from the standpoint of applicability, it raises a set of problems which Boehlje and White [2] and Cocks and Carter [4] failed to realize. The essence of these problems lies with the fact that consumption, which constitutes no part of the objective, plays the role of an income tax, and this leads to a « distorted » solution. The distorsion reveals

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itself in two major ways: first, the internal rate of return will be undervalued; second, the pricing of production factors will be incorrect. As long as the programmed unit is a family firm, this may not pose a grave problem, since the program can be physically implemented regardless of the pricing system. When dealing with a sector or a region of a free economy, however, central imposition of the program is not usually possible, and the guidence is provided by the price system. An incorrect one will hardly serve the purpose.

The objective of this article is to present an applicable scheme for the proper use of the Keynesian consumption function, avoiding incorrect pricing. The core of the argument is a proposition proved in Section B, which yields the correct programming procedure. In section A, we briefly review some results obtained in the absence of consumption, illuminating in the process some points which seem to be of particular interest. Section C reviews an application of the model to the planning of the agricultural sector in a region in Israel.

A. *Inter-Temporal Equilibrium*

A detailed mathematical formulation of the model would involve extremely cumbersome notation. It thus seems easier to describe the model in stages. Consider a period of T years for which a program of production and investment is to be constructed. The economic unit involved can both borrow to finance its operations, and lend if it has no better alternatives, and is interested in maximizing its net worth (equity assets), ω, at the programming horizon. The interest rate charged on money borrowed during year t, i_t, is assumed to exceed the rate paid on deposists, r_t. Since it is assumed that the unit may borow unlimited sums at the going rate, one may view all loans as one-year loans. This approach does not cause any loss of generality and simplifies the exposition.

Problem I is to find non-negative x^t, z^t, $t = 1, 2, \ldots, T$ (T being the programming horizon) which maximize ω subject to

$$(1) \quad \begin{bmatrix} k^{11} & m^{11} & 0 & 0 & \cdots & \cdots & 0 \\ A^{11} & B^{11} & \Phi & \Phi & \cdots & \cdots & 0 \\ k^{21} & m^{21} & k^{22} & m^{22} & \cdots & \cdots & 0 \\ \Phi & B^{21} & A^{22} & B^{22} & \cdots & \cdots & 0 \\ & & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & & & & \\ 0 & m^{T1} & 0 & m^{T2} \,.. & k^{TT} & m^{TT} & 0 \\ \Phi & B^{T1} & \Phi & B^{T2} \,.. & A^{TT} & B^{TT} & 0 \\ 0 & s^1 & 0 & s^2 \,.. & p^T & s^T & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ z^1 \\ x^2 \\ z^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z^T \\ z^T \\ \omega \end{bmatrix} \leqq \begin{bmatrix} \mu_1 \\ q^1 \\ \mu_2 \\ q^2 \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \mu^T \\ q^T \\ \Omega \end{bmatrix}$$

In (1), matrices are denoted by capital letters, vectors by lower case letters and scalars by Greek letters; a null matrix and null vector of any dimension are denoted by Φ and o , respectively, and the real zero is 0 ; transposition indicators are omitted, since only the last column of the matrix in (1), the variable vector and the right hand side, contain column vectors.

The matrix in (1), except for the last column, is composed of production, financial and investment activities. The set of columns having k^{tt} as its first non-zero row, describes production and financial operations, which can be undertaken in year t . Here, k^{tt} is the vector of cash requirements, A^{tt} is the matrix of input coefficients of production factors and k^{t+1t} is the revenue vector. The only exception are borrowing activities, which have the element (-1) in k^{tt} and $(1+i_t)$ in k^{t+1t} .

An investment activity is defined as an operation which augments availability of production resources at some time after its initiation. The set of columns having m^{tt} as its first non-zero row, represents investment projects which can commence in year t. The matrix $B^{\tau t}$, $\tau = t , \ldots , T$ will have positive elements for inputs required and negative ones for outputs forthcoming.

We denote by p^T the vector whose elements represent contributions to (if negative) or claims against (if positive) terminal wealth, which result from production or financial operations in year T . Similarly, s^t is the vector of value residues contributed to terminal wealth by investment projects, which started in year t .

The vectors x^t and z^t are, respectively, the levels of activation of production and investment activities. The scalar μ_1 is the initial endowment of cash and q^1 plays the same role for real production factors. For $t > 1$, μ_t and q^t represent *exogenously* supplied resources. For some t , we may have $\mu_t = 0$, while q^t will be, in general, the non-obsolete remainder of q^1 . Finally, Ω is the value of the non-obsolete portion of q^1 at the horizon.

In order to indicate the main properties of the solution to Problem I, we associate with μ_t , Ω and q^t the shadow prices λ_t , λ_{T+1} and u^t , respectively. The solution to Problem I will be denoted by $\{(\bar{x}^t , \bar{z}^t , \bar{\omega} , \bar{\lambda}_t , \bar{u}^t) , t = 1 , \ldots , T , \bar{\lambda}_{T+1}\}$. Also, for convenient reference, we refer to the set of columns in (1) having k^{tt} and m^{tt} as their first non-zero rows, as A^t and B^t , respectively.

1. The shadow price $\bar{\lambda}_t$ is the value to the program of an additional cash unit made available in year t , in terms of terminal wealth. That is, $\bar{\lambda}_t$ is the value of an additional dollar in year t , compounded

to the horizon. Thus, a dollar added at the horizon is worth one dollar. This can be seen, assuming a non-trivial solution, from the fact that at optimum $\overline{\lambda}_{T+1} = 1$.

The annual discount factor ρ_t, which is the program's annual equilibrium rate of return, is computed by

$$(2) \qquad \overline{\rho}_t = \frac{\overline{\lambda}_t}{\overline{\lambda}_{t+1}} - 1 \qquad t = 1, \ldots, T$$

so that

$$(3) \qquad \overline{\lambda}_t = \prod_{\tau = t}^{T} (1 + \overline{\rho}_\tau)$$

Inspection of the dual equations corresponding to B^t shows, that if in year t money is borrowed, $\overline{\rho}_t = i_t$, while if lending takes place, $\overline{\rho}_t = r_t$. In general,

$$(4) \qquad r_t \leq \overline{\rho}_t \leq i_t ,$$

since money can always be lent in the absence of superior activities, or borrowed in unlimited amounts (1).

The same rate of return applies to all activities. For instance, let $a^t{}_j$ be any column of A^t such that $\overline{x}^t{}_j > 0$. Then by duality,

$$k_j{}^{tt}\, \overline{\lambda}_t + a_j{}^{tt}\, \overline{u}^t + k_j{}^{t+1t}\, \overline{\lambda}_{t+1} = 0 .$$

It follows, that

$$(5) \qquad \frac{\overline{\lambda}_t}{\overline{\lambda}_{t+1}} = \frac{- k_j{}^{t+1t} - (1/\overline{\lambda}_{t+1})\,(a_j{}^{tt}\, \overline{u}^t)}{k_j{}^{tt}}$$

The right hand side of (5) is a « natural » definition of the rate of return, while the left hand is, by (2), $1 + \overline{\rho}_t$. Note the division by $\overline{\lambda}_{t+1}$ in the numerator of (5). It is a result of the fact that $\overline{u}^t$ is a vector of compounded shadow prices, and λ_{t+1} rediscounts them to year t.

The rates of interest discussed above are money rates. It is, of course, most convenient to take money as the numeraire. From a strict theoretical point of view, however, this is an arbitrary convention. Rates of interest can be calculated in terms of any good. The alternative rates will not, in general, be equal to the money rate (2).

(1) See Hirshleifer [6].
(2) See Malinvaud [9 p. 146].

The interpretation we gave to our model is dynamic, intertemporal. However, one could conceive of an instantaneous programming problem that will be formally identical to ours. It is only the meaning we attributed to the magnitudes and indices that made the model into an inter-temporal program. Bearing this point in mind helps to overcome some of the difficulties in interpretation.

3. It is very rare that long range programs are followed to the last year. In most cases changing economic circumstances will force replanning. Multiperiod programming is undertaken in order to optimize short-rum actions, taking into account their long-run effect. It should therefore be interesting to investigate the effects of changing the length of the time span to which the program refers. To this end a comparison between a single-period and a multiperiod analysis is carried out. It will be shown that if the parameters in the single-period program are appropriately specified. the single-period program will yield a solution identical to the first peiiod of the multiperiod solution.

Consider the problem of finding non-negative x^1, z^1, which maximize $\omega\lambda_1$ subject to

$$(6) \qquad \begin{bmatrix} k^{11} & m^{11} & 0 \\ A^{11} & B^{11} & 0 \\ p & s & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ z^1 \\ \omega_1 \end{bmatrix} \leqq \begin{bmatrix} \mu_1 \\ q^1 \\ \Omega_1 \end{bmatrix}$$

Here, ω_1 is equity wealth at the end of the first year; Ω_1 is the end-of-year 1 value of the non-obsolete part of q^1; p and s are equivalent to p^T and s^T in (1). Letting $(x^{1*}, z^{1*}, \omega^*_1)$ be the optimal solution to the first year subproblem of Problem I, one can prove easily that if

$$(7) \qquad \lambda = \bar{\lambda}_2$$

$$(8) \qquad s = \Sigma^T_{t=2} \; (m^{t1} \; \bar{\lambda}_t + \tilde{B}^{t1} \; \bar{u}^t) + s^1$$

where $\backsim$ denotes transposition, then

$$(x^{1*}, z^{1*}) = (\bar{x}^1, \bar{z}^1)$$

The last finding amounts to a restatement of the recursive nature of dynamic production processes: (3) given the appropriate prices, the economy can move from one period to the next, optimizing short run behavior within each period and, at the same time, following the long-rum optimal path. This is also a demonstration of Pontryagin's Maximum Principle (see e. g. Dorfman [4]).

(3) In discussing this point we benefited considerably from Jorgenson's « Lecture Notes on Capital Theory », Hebrew University, 1967 (mimeo).

The above discussion has practical implications centered on the terminal value of assets. These values are, as is evident from (8), the streams of income generated by these assets beyond the programming horizon. It also follows from (8) that in order to evaluate these streams correctly, one has to use the elements of $\bar{u}^t$ which, by (5), contain a compunding factor. For years beyond the horizon the correct compounding factor is not known and can only be guessed. One is thus liable to introduce mistakes which are compounded by each other. Hence, it stands to reasson that the closer the programming horizon — the larger the error, and hence its effects on the first year solution are more profound.

B. *Consumption*

Consumption in year t, c_t, is introduced via the Keynesian consumption function,

$$c_t = \alpha + \beta y_t , \qquad t = 1, \dots, T - 1 \tag{9}$$

where y_t is income in year t, to be defined below, and α, $0 < \beta < 1$, are parameters. Consumption in year T is not explicitly introduced — it is contained in the terminal wealth.

In order to formulate the new programming problem, the elemnts comprising net income, y_t, have to be defined. Two principal categories of income — cash income from production and financial activities, and income in the form of appreciation of assets — are distinguished. Income vectors in the first category, g^t, are defined by

$$g^t = - k^{t+1t} - k^{tt} \tag{10}$$

Note, that for a lending activity j, $g^t_j = r_t$, while if j is a borrowing activity, $g^t_j = - i_t$. The rationale underlying the definition in (10) is established as follows: assume for the sake of exposition that $\bar{x}^t > 0$. Then by duality,

$$\mathrm{k}^{tt} \bar{\lambda}_t + \breve{A}^{tt} \bar{u}^t + \mathrm{k}^{t+1t} \bar{\lambda}_{t+1} = \underline{0} \tag{11}$$

Using the definition in (10), it follows from (3) and (11) that

$$g^t = [\breve{A}^{tt} \bar{u}^t / \prod_{\tau = t+1}^{T} (1 + \bar{\rho}_\tau)] + \bar{\rho}_t k^{tt} \tag{12}$$

The right-hand side of (12) is the current value return to equity assets employed in A^t, and thus constitutes a natural definition of net income.

Income due to appreciation (if positive) or depreciation (if negative) of assets, is the difference in the value of these assets between successive years. Let $w^{\tau t}$, $\tau = t, \dots T$, be the value-vector of assets whose construction began in year t. Then $w^{\tau t}$ is defined

(13) $$w^{\tau t} = -(1/\bar{\lambda}_{\tau+1})\,[\sum_{\theta=\tau+1}^{T} (m^{\theta t}\,\bar{\lambda}_\theta + \breve{B}^{\theta t}\,\bar{u}^\theta) + s^t]$$

In view of the role played by $\bar{\lambda}_t$ and $\bar{u}^t$, (13) is the net stream of benefits produced by investment projects from τ onwards, discounted to τ.

Appreciation (or depreciation) is now defined by

(14) $$v^{\tau t} = w^{\tau t} - w^{\tau-1 t} - m^{\tau t}, \qquad \tau = 1, \dots, T \qquad w^{ot} = \underline{0}$$

In order to establish the logic of the definition in (14), assume $\bar{z}^t > 0$. Then

(15) $$\sum_{\theta=t}^{T} (m^{\theta t}\,\bar{\lambda}_\theta + \breve{B}^{\theta t}\,\bar{u}^\theta) + s^t = \underline{0}$$

Dividing (15) by $\bar{\lambda}_t$ and $\bar{\lambda}_{\tau+1}$ and using the resulting equations together with (13) and (14), one obtains

(16) $$v^{\tau t} = (1/\bar{\lambda}_{\tau+1})\,\breve{B}^{\tau t}\,\bar{u}^\tau + \bar{\rho}_\tau\,m^{\tau t} + (\bar{\rho}_\tau/\bar{\lambda}_\tau)\sum_{\theta=t}^{\tau-1} (m^{\theta t}\bar{\lambda}_\theta + \breve{B}^{\theta t}\bar{u}^\theta)$$

The first term on the right hand side of (16) represents the change in value due to truncation of the income stream by one period; the second term is the return on cash investments during year τ and the third term expresses returns to equity assets invested in the project up to, and excluding, year τ.

As equation (16) indicates, appreciation and depreciation are functions of the optimal solution and thus cannot be known when the problem is set up. This fact prohibits us from direct application of the proposition proved below which, nevertheless, is of economic interest and quite useful, as will be indicated.

Rewriting (9) in the constraint form

(17) $$-c_t + \beta y_t \leq -\alpha,$$

Problem II is to find non-negative x^t, z^t, c^t which maximize

(18) $$f = \sum_{t=1}^{T-1} \delta_t\, c_t + \omega$$

subject to

$$
(19)\qquad
\begin{bmatrix}
\bar{k}^{11} & m^{11} & 0 & 0 & 0 & \cdots & & & & 0 \\
A^{11} & B^{11} & 0 & \Phi & \Phi & \cdots & & & & 0 \\
\beta g^{1} & \beta v^{11} & -1 & 0 & 0 & \cdots & & & & 0 \\
k^{21} & m^{21} & 1 & \bar{k}^{22} & \bar{m}^{22} & \cdots & & & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \beta v^{T-1,1} & 0 & 0 & \beta v^{T-1,2} & \cdots & -1 & 0 & 0 & 0 \\
0 & m^{T1} & 0 & 0 & m^{T2} & \cdots & 1 & \bar{k}^{TT} & \bar{m}^{TT} & 0 \\
\Phi & B^{T1} & 0 & \Phi & B^{T2} & \cdots & 0 & A^{TT} & B^{TT} & 0 \\
0 & s^{1} & 0 & 0 & s^{2} & \cdots & 0 & p^{T} & s^{T} & 1
\end{bmatrix}
\begin{bmatrix}
x^{1} \\ z^{1} \\ c_{1} \\ x^{2} \\ \cdot \\ \cdot \\ \cdot \\ c_{T-1} \\ x^{T} \\ z^{T} \\ \omega
\end{bmatrix}
\leqq
\begin{bmatrix}
\bar{\mu}_{1} \\ q^{1} \\ -\alpha \\ \mu^{2} \\ \cdot \\ \cdot \\ \cdot \\ \mu_{T} \\ q^{T} \\ \Omega
\end{bmatrix}
$$

where δ_t are as yet unspecified parameters.

Let Problem IIa be Problem II with

$$(20)\qquad \delta_t = 0 \qquad\qquad t = 1, \ldots, T-1$$

and denote the optimal solution, and the shadow prices associated with it, by $\{x^t, z^t, c^t, \omega\}$ and $\{\lambda_t, u^t, \eta_t\}$, respectively, where $\{\eta_t\}$ are the shadow prices of the consumption constraints. Assuming

$$(21)\qquad c_t > 0, \qquad\qquad t = 1, \ldots, T-1$$

it follows that

$$(22)\qquad \eta_t > 0,$$

since consumption plays the role of a mere income tax. This imposes a burden whose magnitude, per marginal dollar, is given by the values of η_t. The distortion which results from the imposition of consumption is best reflected in the fact, that the internal rate of return to which the system now adjusts itself is *net* of consumption expenditures. This can be best seen if one assumes, for simplicity, that in year t borrowing takes place. Then, by duality, we have instead of (2)

$$\frac{\lambda_t}{\lambda_{t+1}} = 1 + (1 - \beta)\, i_t \equiv 1 + \rho_t$$

so that $\rho_t = (1 - \beta)\, \bar{\rho}_t < \bar{\rho}_t$. As a result, one has instead of (5)

$$(5a)\qquad \frac{\lambda_t}{\lambda_{t+1}} = \frac{-k_j^{t+1t} - (1/\lambda_{t+1})\,(a_j^{tt}\, u^t)}{k_j^{tt}} - \beta\, \frac{g_j^{t}}{k_j^{tt}}$$

where the last term in (5a) represents the effect of consumption on the internal rate of return. It indicates, that the solution « tries to avoid » activities with a high ratio of income (or revenue) to expenditure.

The elimination of these difficulties can be achieved only through a « correct » selection of the δ_t values. Technically, the selection procedure can be effected via a search scheme involving parametric programming. The search procedure is terminated when a set $\{\delta_t^*\}$ and a corresponding primal-dual solution $\{x^{t*}, z^{t*}, c^{t*}, \omega^*, \lambda_t^*, u^{t*}, \eta_t^*\}$ are reached such that

(23) $$\delta^*_t = \lambda^*_{t+1}, \qquad t = 1, \ldots, T-1$$

(24) $$-c^*_t + \beta y^*_t = -\alpha, \qquad t = 1, \ldots, T-1$$

Looking at the consumption column in (19), it is obvious that

(25) $$-\eta^*_t + \lambda^*_{t+1} \geq \delta_t, \qquad t = 1, \ldots, T-1$$

Thus, an immediate consequence of (23) is

(26) $$\eta^*_t = 0, \qquad t = 1, \ldots, T-1$$

Equations (24) and (26) imply that we are looking for a so-called degenerate solution.

Under (23) one can show, that if $\{\bar{\lambda}_t, \bar{u}^t\}$ in (16) is replaced with $\{\lambda^*_t, u^{t*}\}$, then (3) and (5) hold for λ^*_t and by (26) consumption is no longer a burden to the system.

Before continuing with the analysis, some remarks are in order. As is well known, the optimum quantities consumed by an individual who maximizes a Fisherian utility function are such that the marginal rates of substitution in consumption between successive years equal the respective marginal rates of return to assets. Our solution is thus consistent with the individual's equilibrium. This consistency should, however, be put in the appropriate perspective: the Keynesian consumption function *cannot* be derived from a general Fisherian utility function. That is, there is no theoretical basis for suggesting that individual consumption behavior can be described by the Keynesian function. On the other hand, that function does describerather well aggregate behavior and the weights attached to consumptlon outlays in our solution, which are derived from aggregate behavior, also satisfy the optimality condition for the individual. These weights, the δ_t values, are equal, by (23), to the marginal contributions of funds reinvested in the program (1).

It is shown in [7], that the search procedure referred to above is finite. Practically speaking, however, it may take considerable time

(1) Note, that we have arrived at a BQ-type model, except that now the coefficients in the « utility function » are based on the observable consumption function.

and be quite expensive. Thus, the objective of the analysis from here on is to suggest instead a simpler procedure. It is based on the relation between the solution of Problem II*a* and the solution which satisfies (26), a relation which is derived below, avoiding the tedious search procedure.

Suppose, then, that Problem II*a* has been solved under the stipulation that in (16), $\{\bar{\lambda}_t \, \bar{u}_t\}$ is replaced with $\{\lambda_t^* \, u^{t*}\}$; i.e., it is assumed for the time being that the sought solution values are known in advance, so that appreciation and depreciation can be calculated without errors. Let D be the basic matrix of the solution. Noting that the problem is cast in the so-called revised form, assume for convenience that the first row of D^{-1} is the « pricing vector » (the shadow price vector). Further, let $\breve{h} = (1 \; \lambda_1 \; \breve{u}^1 \; \lambda_2 \; \breve{u}^2 \ldots \lambda_T \; \breve{u}^T)$, $\breve{e} = (\eta_1 \; \eta_2 \ldots \eta_{T-1})$. Then D^{-1} may be written as

$$D^{-1} = \begin{bmatrix} \breve{h} & \breve{e} \\ D^{11} & D^{12} \\ D^{21} & D^{22} \end{bmatrix}$$

where D^{22} contains the elements common to the consumption rows and columns.

Next, let $\breve{d} = (\delta_1 \; \delta_2 \ldots \delta_{T-1})$ and compute d^o from

$$\breve{d}^o D^{22} = -\breve{e} \tag{27}$$

Insert $d = d^o$ in the objective (18), without resolving the problem. This will give rise to a new set of shadow prices, $(h^o \, e^o)$. For $d = \underline{0}$, we have

$$(\breve{h} \; \breve{e}) = (1 \; \underline{\breve{0}} \; \underline{\breve{0}}) \; D^{-1}, \tag{28}$$

and for $d = d^o$,

$$(\breve{h}^o \; \breve{e}^o) = (1 \; \underline{\breve{0}} \; \breve{d}^o) \; D^{-1} \tag{29}$$

From (28) and (29),

$$(\breve{h}^o \; \breve{e}^o) = (\breve{h} \; \breve{e}) + (0 \; \underline{\breve{0}} \; \breve{d}^o) \; D^{-1} \tag{30}$$

which implies, together with (27),

$$e^o = \underline{0}, \tag{31}$$

which is equivalent to (26). That (24) is satisfied by c_t is obvious. Moreover, by (21) and (31)

$$\delta^o{}_t = \lambda^o{}_{t+1} \tag{32}$$

which is equivalent to (23). It is thus evident, that if the solution of Problem II*a* is optimal with respect to d^o, then (30) indeed provides a simple way for calculating the correct shadow prices. The optimality of the solution is established below.

Equation (30) gives us the usual sensitivity analysis technique to solve for h^o. It can be verified, that in this case the relation between h and h^o is given by

$$(33) \qquad \lambda_t = \lambda^o{}_t \prod_{\tau = t}^{T-1} \left[1 - \frac{\beta \rho_\tau{}^o}{1 + \rho_t{}^o}\right] \qquad t = 1, \ldots, T-1$$

$$(34) \qquad u^t = (1 - \beta)\; u^{to} \prod_{\tau = t+1}^{T-1} \left[1 - \frac{\beta \rho_t{}^o}{1 + \rho_\tau{}^o}\right] \qquad t = 1, \ldots, T-2$$

$$(35) \qquad \lambda_T = \lambda^o{}_T$$

$$u^T = u^{To}$$

$$(36) \qquad u^{T-1} = (1 - \beta)\; u^{(T-1)o}$$

where $\rho^o{}_t$ is computed from (2) with $\lambda_t = \lambda^o{}_t$.

Proposition: If $v^{\tau t}$ in (16) is computed using $\{\lambda^*{}_t, u^{t*}\}$, then D is an optimal basis for Problem II with $d = d^*$.

Proof: To prove the proposition, one must show that the pricing vector $(h^o\ e^o)$ and its dot product with any column of the matrix in (19), are non-negative. Starting with the former, it follows from (33) that

$$(37) \qquad \lambda_t = \lambda^o{}_t \left(1 - \frac{\beta \rho^o{}_t}{1 + \rho^o{}_t}\right) \frac{\lambda_{t+1}}{\lambda^o{}_{t+1}}$$

From (2) and (37),

$$\lambda_t = [(1 - \beta)\; \lambda^o{}_t + \beta \lambda^o{}_{t+1}]\; \frac{\lambda_{t+1}}{\lambda^o{}_{t+1}}$$

which gives, rarranging terms,

$$(38) \qquad (1 - \beta)\; \frac{\lambda^o{}_t}{\lambda^o{}_{t+1}} = -\beta + \frac{\lambda_t}{\lambda_{t+1}}$$

Taking now the lending activity of year t and noting that in view of (21) $\eta_t = \lambda_{t+1}$, we have

$$(39) \qquad \frac{\lambda_t}{\lambda_{t+1}} \geq 1 + (1 - \beta)\, r_t > 1$$

Letting ρ_t satisfy

(40) $$\frac{\lambda_t}{\lambda_{t+1}} = 1 + (1 - \beta)\,\rho_t\,,$$

it follows from (38), (39) and (40) that

(41) $$\frac{\lambda^o{}_t}{\lambda^o{}_{t+1}} = 1 + \rho_t \equiv 1 + \rho^o{}_t > 1$$

This implies, together with $\beta < 1$,

(42) $$1 - \frac{\beta\rho^o{}_t}{1 + \rho^o{}_t} > 0$$

Using (35), (42) and applying (33) recursively, it follows that

(43) $$\lambda^o{}_t > 0\,, \qquad\qquad t = 1\,, \ldots\,, T$$

Using now (36), (42) and applying (34) recursively we also have

(44) $$u^{to} > 0\,, \qquad\qquad t = 1\,, \ldots\,, T$$

which, recalling (31), concludes the first part of the proof.

For any submatrix A^t, we have by the optimality of the solution to Problem II*a*

(45) $$\xi\,(x^t) \equiv k^{tt}\,\lambda_t + \widetilde{A}^{tt}\,u^t - \beta\,(k^{t+1t} + k^{tt})\,\lambda_{t+1} + k^{t+1t}\,\lambda_{t+1} \geqq 0$$

Using (31), (33) and (34) together with (45), we find

(46) $$k^{tt}\,\lambda^o{}_t + \widetilde{A}^{tt}\,u^{to} + k^{t+1t}\,\lambda^o{}_{t+1} = (1 - \beta)\;\xi\,(x^t) \geqq 0\,.$$

By the same reasoning underlying (45), we have for any submatrix B^t

(47) $$\xi\,(z^t) \equiv \sum_{\tau = t}^{T} (m^{\tau t}\,\lambda_\tau + \widetilde{B}^{\tau t}\,u^\tau) + \beta \sum_{\tau = t}^{T-1} v^{\tau t}\,\lambda_{\tau+1} + s^t \geqq 0$$

Using now (3), (16), (33), (34), (35), (36) and (46), it is not difficult to verify that

(48) $$\xi\,(z^t) = \sum_{\tau = t}^{T} (m^{\tau t}\,\lambda^o{}_\tau + \widetilde{B}^{\tau t}\,u^{\tau o}) + s^t\,,$$

and since $\xi\,(z^t)$ is non-negative, so is the right-hand-side of (48). Q.E.D.

The economic rationale underlying the property just established is, that the combination of activities which contributes most to terminal wealth, contributes most to consumption and terminal wealth provided these contributions are correctly measured.

Correct computations of appreciation and depreciation, necessary for the validity and direct applicability of the above proposition, cannot be effected at the formulation stage of the problem, since they involve solution values. In application, net income elements are computed by common accounting procedures, and will usually differ from the correct values. This will result in a situation in which investment activities with lower appreciation (higher depreciation) values will contribute more to the terminal wealth than to consumption as compared with investment actvities with higher appreciation or lower depreciation values. Thus, the primal solution to Problem II*a* will not, in general, remain unaffected by the insertion of d^o in (18). It is, however, our experience that applying the proposition, despite the practical shortcomings, reduces very considerably the computations involved in achieving a solution satisfying (23) and (24). This procedure was applied in the practical example illustrated in the next Section.

C. *Illustration*

The model discussed in Section *B* was applied to the agricultural sector of a region in nothern Israel. The program spans ten years and comprises a total of 360 production and investment activities, 9 consumption, 55 financial and 60 other activities - 484 in all. The constraint set consists of 459 constraints, of which 190 relate to limited production factors.

Some of the results are given in Tables 1 and 2. Table 1 contains part of the solution to Problem II*a*. From this solution d^o was computed in the manner indicated by (27) and Problem II was resolved with $d = d^o$. Although the second solution did not satisfy (31), the elements of e^o were so small that no further iterations seemed required. As could be expected, the two primal solutions are somewhat different, owing to errors in appreciation and depreciation of assets. In particular, most consumption elements in the second solution are higher than the corresponding ones in the first solution.

A few comments regarding the numerical results are in order. First, the high rates of interest, up to 13%, are not too high in a country where annual inflation rates are between 5% and 10%. Next, the impression that consumption is being programmed should not stay unamended. What is programmed is the amount of funds diverted from the production to the consumption sector. Some of these funds may be invested in durable consumer goods and thus need not increase

TABLE 1. - *Solution of Problem IIa.*

Year t	λ_t	η_t	c_t IL. 10^6	Sugar beet acres	Planting pears acres	One year credit IL. '000
1	1.258	1.252	16.057	650	495	—
2	1.252	1.236	17.234	650	173	48
3	1.236	1.221	16.772	650	142	—
4	1.221	1.205	15.974	650	108	433
5	1.205	1.190	15.100	650	23	2674
6	1.190	1.175	16.717	553	—	5107
7	1.175	1.160	17.232	354	—	5806
8	1.160	1.144	17.712	260	—	6335
9	1.144	1.130	18.657	170	—	4558
10	1.130	—	—	—	—	969

TABLE 2. - *Second solution of Problem II.*

Year t	δ^0_t	λ^0_t	η^0_t	ρ^0_t	c^0_t IL. 10^6	Sugar beet acres	Planting pears acres	One year credit IL. '000
1	2.977	3.150	.024	.058	16.525	650	650	—
2	2.634	3.001	.024	.13	17.146	650	650	1051
3	2.352	2.658	0.0	.12	16.386	650	625	1271
4	2.081	2.352	.001	.13	15.354	650	100	3767
5	1.842	2.082	0.0	.13	16.776	650	66	7957
6	1.630	1.842	0.0	.13	17.872	650	—	15262
7	1.443	1.630	0.0	.13	19.422	650	—	21123
8	1.277	1.443	0.0	.13	20.264	650	—	17894
9	1.129	1.277	.001	.13	20.871	649	—	13986
10	—	1.130	—	.13	—	—	—	9496

monotonically. This clearly merits a separate treatment, but falls without the scope of the present discussion.

Related to the last remark is the relatively low rate of annual consumption increase which proceeds at an average pace of 3% in the second solution. It is particularly low in Israel, where a rate of 8% is not uncommon. This is probably due to the fact that we did not take into account technological progress.

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