האוניברסיטה העברית בירושלים

The Hebrew University of Jerusalem

המרכז למחקר בכלכלה חקלאית

The Center for Agricultural Economic Research

המחלקה לכלכלה חקלאית ומנהל

The Department of Agricultural Economics and Management

Discussion Paper No. 9.14

A simple method for analyzing optimal steady states in multi-dimensional resource models

by

Yacov Tsur and Amos Zemel

Papers by members of the Department can be found in their home sites:

מאמרים של חברי המחלקה נמצאים גם באתרי הבית שלהם:

http://departments.agri.huji.ac.il/economics/index.html

P.O. Box 12, Rehovot 76100, Israel

ת.ד. 12, רחובות 76100

A simple method for analyzing optimal steady states in multi-dimensional resource models

Yacov Tsur* Amos Zemel◇

December 26, 2014

Abstract

We study optimal steady states in multi-dimensional resource models with minimal assumptions on the underlying processes. A function $L : \mathbf{X} \mapsto \mathbb{R}^n$ is defined over the set of feasible states $\mathbf{X} \subseteq \mathbb{R}^n$ in terms of the model's primitives. This function and its derivatives are then used to obtain necessary conditions regarding the location and stability of optimal steady state candidates. Examples illustrating the applicability of the method show that it can reduce the list of optimal steady state candidates to a singleton in a variety of resource problems.

Keywords: multiple states, dynamic optimization, steady states, stability, resource management.

JEL classification: C61, C62, Q20, Q25, Q30

*Department of Agricultural Economics and Management and the Center for Agricultural Economic Research, The Hebrew University of Jerusalem, POB 12, Rehovot 76100, Israel (tsur@agri.huji.ac.il).

◇Department of Solar Energy and Environmental Physics, The Jacob Blaustein Institutes for Desert Research, Ben Gurion University of the Negev, Sede Boker Campus 8499000, Israel (amos@bgu.ac.il).

1 Introduction

Optimal resource policies often converge towards steady states, in which case locating the steady state is of prime importance for several reasons. First, steady states parsimoniously characterize the optimal policy in the long run. Second, analyzing the dependence of the steady states on various parameters of the model provides insights on long-run tradeoffs. Finally, the location of optimal steady states can help deriving the full dynamics of optimal policies, because the end-conditions they provide are often easier to use than the corresponding infinite horizon transversality conditions. Building on the single-state method of Tsur and Zemel (2001, 2014), we develop in this work a simple method to locate optimal steady-state candidates in multi-dimensional resource models and study their stability via a function that is readily obtained from the model's primitives.

The long-run behavior of economic processes was thoroughly investigated quite a while ago in the context of multi-sector economic growth (see Kurz 1968, Levhari and Liviatan 1972, Cass and Shell 1976, Rockafellar 1976, Brock and Scheinkman 1976, 1977, Sorger 1989, and works they cite). In that context it was natural to make certain concavity-convexity assumptions (e.g., of utility functions and of production technologies), which greatly facilitated the analysis of properties such as the global stability of the steady states. Such assumptions, however, are quite restrictive and as a result this body of research has not been often used outside the realm of economic growth. A case in point is the area of resource economics, where non-convexities are pervasive (Dasgupta and Mäler 2003).

The method developed here requires minimal assumptions on the underlying processes. It is based on a function $L : \mathbf{X} \mapsto \mathbb{R}^n$ defined over the set of feasible states $\mathbf{X} \subseteq \mathbb{R}^n$ in terms of the model's (primitive) functions. This function and its derivatives are used to identify the location of optimal steady states and to study the (local) stability of each candidate. The simplicity and relative ease of use of this method are due to several features. First, the analysis is carried out within the $n-$dimensional state space rather than the $2n-$dimensional state-costate space. Second, the L-function is readily obtained in terms of the model's primitives without resorting to the optimality conditions. Moreover, the properties derived require weak curvature assumptions, formulated in terms of the ordinary Hamiltonian rather than the maximized Hamiltonian. In particular, we require the Hamiltonian to be concave with respect to the action variables only (rather than with respect to the state and action variables jointly).

Obviously, these advantages must come at a cost, and the local analysis embodied in the L-function method can give rise to necessary conditions only. This means that the method can disqualify unsuitable candidates for an optimal steady state but, in general, cannot establish that a certain state is the eventual target of the optimal state process. Nor can it identify cyclical behavior, such as in Benhabib and Nishimura (1979), Dockner and Feichtinger (1991), Feichtinger et al. (1994) or Wirl (1992, 1995). Nonetheless, it provides useful information, which narrows the list of optimal steady state candidates. Indeed, the two-state examples presented in Section 5 show that the method can reduce the list of candidates to a singleton in a variety of resource management problems.

2 The model

We consider the n-state resource model:

$$v(X_0) = \max_{\{C(t)\}} \int_0^\infty f(X(t), C(t)) e^{-\rho t} dt \tag{2.1a}$$

subject to

$$\dot{X}(t) = G(X(t), C(t)) \tag{2.1b}$$

$$X(0) = X_0 \,, \tag{2.1c}$$

where the constant $\rho > 0$ is the discount rate, $X = (x_1, x_2, \ldots, x_n)' \in \mathbf{X} \subseteq \mathbb{R}^n$ denotes the vector of state variables, $C = (c_1, c_2, \ldots, c_n)' \in \mathbf{C} \subseteq \mathbb{R}^n$ is the vector of actions (controls), and $\mathbf{X}$ and $\mathbf{C}$ are the admissible state and action sets, respectively (the prime denotes transpose). For the sake of concreteness, we specify the admissible state set as $\mathbf{X} \equiv \prod_{i=1}^n [\underline{x}_i, \bar{x}_i]$ for some given constants $\underline{x}_i < \bar{x}_i$, $i = 1, 2, ..., n$. The function $f : \mathbf{X} \times \mathbf{C} \mapsto \mathbb{R}^1$ is the instantaneous benefit (utility) function, and the vector function $G = (g_1, g_2, \ldots, g_n)'$, with $g_i : \mathbf{X} \times \mathbf{C} \mapsto \mathbb{R}^1$, represents the time evolution of the states x_i, $i = 1, 2, \ldots, n$. It is assumed that all functions are sufficiently smooth and that problem (2.1) admits an optimal policy, thus $v : \mathbf{X} \mapsto \mathbb{R}^1$ exists.

We further assume that for each state there exists at least one influential control. Put differently, the Jacobian of G with respect to C,

$$J_C^G(X, C) = \begin{pmatrix} g_{1\,c_1} & g_{1\,c_2} \cdots & g_{1\,c_n} \\ g_{2\,c_1} & g_{2\,c_2} \cdots & g_{2\,c_n} \\ & \vdots & \\ g_{n\,c_1} & g_{n\,c_2} \cdots & g_{n\,c_n} \end{pmatrix},$$

is non-singular.[1] More precisely, we assume that the determinant of the Jacobian is bounded away from zero:

[1] Variable subscripts denote partial derivatives, e.g., $g_{i\,c_j} \equiv \partial g_i(X, C)/\partial c_j$.

Assumption 1. $|\det(J_C^G(X,C))| \geq \xi > 0 \; \forall \, (X,C) \in \boldsymbol{X} \times \boldsymbol{C}$.

Define (implicitly) the vector function $M = (m_1, m_2, \ldots, m_n)' : \mathbf{X} \mapsto \mathbf{C}$, by

$$G(X, M(X)) = 0. \tag{2.2}$$

When the system is at the state X, choosing the steady state policy $C = M(X)$ maintains the state X indefinitely and yields the payoff

$$W(X) \equiv f(X, M(X))/\rho \leq v(X), \tag{2.3}$$

where the inequality holds as an equality only at the optimal steady state. It is assumed that the steady state policy is feasible for all $X \in \mathbf{X}$, i.e., $M(X) \in \mathbf{C}$. In general (without additional structure) there is no assurance that the optimal state process converges to a steady state and when it does there could be several steady state candidates (depending, inter alia, on the point of departure X_0). In the next section we study the location of optimal steady-state candidates.

3 Optimal steady state candidates

Differentiating (2.2) with respect to X gives $J_X^G + J_C^G J_X^M = 0$ or

$$J_X^M(X) = -[J_C^G(X, M(X))]^{-1} J_X^G(X, M(X)), \tag{3.1}$$

where $J_X^G(X,C)$ denotes the Jacobian matrix of $G(X,C)$ with respect to X and $J_X^M(X)$ is the Jacobian matrix of $M(X)$. Since the inverse matrix $[J_C^G(X, M(X))]^{-1}$ exists (Assumption 1), $J_X^M(X)$ is well defined. Using (3.1) we obtain

$$\begin{aligned} &\rho W_X(X) = f_X(X, M(X)) + [J_X^M(X)]' f_C(X, M(X)) \; = \\ &f_X(X, M(X)) - [J_X^G(X, M(X))]'[J_C^G(X, M(X))]'^{-1} f_C(X, M(X)), \end{aligned} \tag{3.2}$$

where $W_X(X) \equiv \nabla_X W(X)$ and $f_X(X,C) \equiv \nabla_X f(X,C)$ are the n-vectors of partial derivatives of $W(X)$ and $f(X,C)$ with respect to X, respectively, and $f_C(X,C) \equiv \nabla_C f(X,C)$ is the vector of partial derivatives of $f(X,C)$ with respect to C.

Define the vector function $L : \mathbf{X} \mapsto \mathbb{R}^n$ by

$$L(X) \equiv \begin{pmatrix} l_1(X) \\ l_2(X) \\ \vdots \\ l_n(X) \end{pmatrix} = \rho \left\{ [J_C^G(X, M(X))]'^{-1} f_C(X, M(X)) + W_X(X) \right\} =$$
$$\left(\rho I_n - [J_X^G(X, M(X))]'\right) [J_C^G(X, M(X))]'^{-1} f_C(X, M(X)) + f_X(X, M(X)) \tag{3.3}$$

where the second equality follows from (3.2).[2] Let $\hat{X} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)' \in \mathbf{X}$ be an optimal steady state. We say that $\hat{X}$ is *unconstrained* if $\hat{X}$ remains an optimal steady state when the admissible state set $\mathbf{X}$ is slightly enlarged.[3] The following property identifies optimal steady state candidates:

Proposition 1. *(i) If $\hat{X}$ is unconstrained, then $L(\hat{X}) = 0$.*
(ii) If $\hat{x}_i = \bar{x}_i$ for some i, then $l_i(\hat{X}) \geq 0$.
(iii) If $\hat{x}_i = \underline{x}_i$ for some i, then $l_i(\hat{X}) \leq 0$.

Proof. For any $X \in \mathbf{X}$, we compare the payoff $W(X)$ obtained under the steady state policy $C = M(X)$ with the payoff obtained from a small feasible variation of this policy. If the variation policy yields a payoff that exceeds $W(X)$, then the steady-state policy is not optimal at X and this state does not qualify as an optimal steady state. For small $\varepsilon > 0$ and $\Delta = (\delta_1, \delta_2, \ldots, \delta_n)'$,

[2] It is easy to see how the two forms of $L(\cdot)$ in (3.3) extend the single-state $L(\cdot)$, defined in equations (2.7)-(2.8) of Tsur and Zemel (2014, p. 167), to multi-state models.

[3] If $\hat{X}$ lies in the interior of $\mathbf{X}$, then $\hat{X}$ is unconstrained, but the converse is not always true, as unconstrained $\hat{X}$ can fall on a boundary (where $\hat{X} \in \partial\mathbf{X}$). Constrained steady states, on the other hand, must fall on a boundary.

the variation policy is defined by

$$C^{\varepsilon\Delta}(t) \equiv \begin{cases} M(X) + [J_C^G(X, M(X))]^{-1}\Delta & \text{if } t < \varepsilon \\ M(X(\varepsilon)) & \text{if } t \geq \varepsilon \end{cases}.$$

While $t < \varepsilon$, $C^{\varepsilon\Delta}(t)$ deviates slightly from the steady-state policy $C = M(X)$, then it enters a steady state at $X(\varepsilon)$. During the first period $t \in [0, \varepsilon)$, $\dot{X} = G(X, M(X)) + J_C^G(X, M(X))[J_C^G(X, M(X))]^{-1}\Delta + o(\delta) = \Delta + o(\delta)$, which brings the state at $t = \varepsilon$ to $X(\varepsilon) = X + \varepsilon\Delta + o(\varepsilon\delta)$.[4]

The contribution to the objective under the variation policy $C^{\varepsilon\Delta}(t)$ during $t < \varepsilon$ is evaluated, up to $o(\varepsilon\delta)$ terms, by

$$\int_0^\varepsilon f\left(X(t), M(X) + [J_C^G(X, M(X))]^{-1}\Delta\right) e^{-\rho t} dt =$$
$$= \int_0^\varepsilon \rho W(X) e^{-\rho t} dt + [f_C(X, M(X))]'[J_C^G(X, M(X))]^{-1}[\varepsilon\Delta],$$

and the contribution during $t \geq \varepsilon$ is evaluated, up to $o(\varepsilon\delta)$ terms, by

$$\int_\varepsilon^\infty f(X(\varepsilon), M(X(\varepsilon))) e^{-\rho t} dt = \int_\varepsilon^\infty \rho W(X(\varepsilon)) e^{-\rho t} dt =$$
$$= \int_\varepsilon^\infty \rho W(X) e^{-\rho t} dt + [W_X(X)]'[\varepsilon\Delta].$$

Summing the contributions of the two periods gives the payoff $V^{\varepsilon\Delta}(X)$ obtained under the variation policy:

$$V^{\varepsilon\Delta}(X) = W(X) + \left[[J_C^G(X, M(X))]'^{-1} f_C(X, M(X)) + W_X(X)\right]' [\varepsilon\Delta] + o(\varepsilon\delta).$$

Thus, noting (3.3),

$$V^{\varepsilon\Delta}(X) - W(X) = [L(X)]'[\varepsilon\Delta]/\rho + o(\varepsilon\delta).$$

The sign of the elements of Δ can be freely chosen, while $\varepsilon > 0$. Now, if $L(X) \neq 0$ we can set $\Delta = \delta L(X)$, where δ is a small positive constant,

[4] The notation $o(\delta)$ indicates terms satisfying $o(\delta)/\delta \to 0$ as $\delta \to 0$; a vector is $o(\delta)$ when all its elements are $o(\delta)$.

hence $[L(X)]'\Delta > 0$. This implies $V^{\varepsilon\Delta}(X) > W(X)$ and X is not an optimal steady state. Thus, only the roots of $L(\cdot)$ qualify as legitimate candidates for an optimal steady state. The only possible exceptions are the bounds of $\mathbf{X}$. Choosing $\delta_i > 0$ is not feasible at $\bar{x}_i$ because this policy would drive the $x_i(\cdot)$ process outside the feasible domain. It follows that $X = (x_1, x_2, \ldots, \bar{x}_i, \ldots x_n)'$ cannot be excluded as an optimal steady state if $l_i(X) > 0$. A similar argument implies that $X = (x_1, x_2, \ldots, \underline{x}_i, \ldots x_n)'$ cannot be excluded as an optimal steady state if $l_i(X) < 0$. $\square$

That $L(\cdot)$ must vanish at an unconstrained steady state can alternatively be seen as follows. Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)'$ denote the (current-value) costate vector and write the current-value Hamiltonian corresponding to problem (2.1) as

$$H(X, C, \Lambda) = f(X, C) + \Lambda' G(X, C). \tag{3.4}$$

Necessary conditions for an interior optimum include

$$f_C(X, C) + [J_C^G(X, C)]'\Lambda = \mathbf{0} \tag{3.5}$$

and

$$\dot{\Lambda} - \rho\Lambda = -f_X(X, C) - [J_X^G(X, C)]'\Lambda. \tag{3.6}$$

From (3.5)

$$\Lambda = -[J_C^G(X, C)]'^{-1} f_C(X, C) \tag{3.7}$$

holds along the optimal process, which brings (3.6) to the form

$$\dot{\Lambda} = \psi(X, C) \equiv -\left(\rho I_n - [J_X^G(X, C)]'\right)[J_C^G(X, C)]'^{-1} f_C(X, C) - f_X(X, C). \tag{3.8}$$

Evaluating (3.8) at the steady state policy $C = M(X)$ and comparing with (3.3), we see that at an interior steady state $\dot{\Lambda} = -L(\hat{X})$, and since $\dot{\Lambda}$ vanishes at a steady state, $L(\hat{X})$ must vanish as well.

We turn now to derive a stability condition for the steady state candidates identified by Proposition 1.

4 Stability

Tsur and Zemel (2014) have shown that in single-state models the condition $dL(X)/dX < 0$ is necessary for a root of $L(\cdot)$ to be locally stable. We extend this result to multi-state models by deriving a necessary condition for local stability in terms of the Jacobian of $L(\cdot)$. Our approach differs from the standard stability analysis (see, e.g, Rockafellar 1976, Brock and Scheinkman 1976, Cass and Shell 1976) in two respects. First, it is based on the $n \times n$ Jacobian matrix $J_X^L(X)$ of $L(\cdot)$ rather than on the $2n \times 2n$ Jacobian matrix of the modified Hamiltonian system defined in the $2n-$dimensional state-costate space. This distinction greatly simplifies the characterization in actual problems, not only due to the smaller dimension of the Jacobian involved, but also because $L(\cdot)$ and its derivatives are directly obtained from the model's functions without resorting to first order conditions on the optimal trajectories. Second, our approach uses properties of the ordinary Hamiltonian rather than those of the maximized Hamiltonian. As a result, our stability condition holds under weaker assumptions. In particular, concavity of the maximized Hamiltonian with respect to the state (which in turn requires that the Hamiltonian is jointly concave in the state and the control) is replaced in the present approach by concavity of the Hamiltonian with respect to the control only.

Let us briefly review the standard approach for analyzing the stability properties of steady states. The maximized Hamiltonian is defined as

$$H^0(X,\Lambda) \equiv \max_{C\in \mathbf{C}} H(X,C,\Lambda) = H(X,C^0(X,\Lambda),\Lambda), \tag{4.1}$$

where $C^0(X,\Lambda)$ is the optimal action satisfying (3.5). The optimal state and costate processes satisfy the necessary conditions

$$\dot{X} = H^0_\Lambda(X,\Lambda), \tag{4.2a}$$

$$\dot{\Lambda} = \rho\Lambda - H^0_X(X,\Lambda) \tag{4.2b}$$

and the Jacobian corresponding to the modified Hamiltonian system (4.2) is

$$J^0(X,\Lambda) = \begin{pmatrix} H^0_{\Lambda X}(X,\Lambda) & H^0_{\Lambda\Lambda}(X,\Lambda) \\ -H^0_{XX}(X,\Lambda) & \rho I_n - H^0_{X\Lambda}(X,\Lambda) \end{pmatrix}. \tag{4.3}$$

Denote by $(\hat{X},\hat{\Lambda})$ a stationary state of (4.2), where $\dot{X} = \dot{\Lambda} = 0$. This state bears the saddle-point property if $H^0(X,\hat{\Lambda}) \le H^0(\hat{X},\hat{\Lambda}) \le H^0(\hat{X},\Lambda)$ for all (X,Λ) in a neighborhood of $(\hat{X},\hat{\Lambda})$. This property requires some curvature of the maximized Hamiltonian near $(\hat{X},\hat{\Lambda})$, which can be detected by examining the eigenvalues of $J^0(\hat{X},\hat{\Lambda})$ (see, e.g., Kurz 1968, Levhari and Liviatan 1972). For example, Rockafellar (1976) used the condition $\rho^2 < 4\alpha\beta$, where ρ is the discount rate, α is the smallest eigenvalue of $H^0_{\Lambda\Lambda}$ and β is the smallest eigenvalue of $-H^0_{XX}$, while Brock and Scheinkman (1976) required the "curvature matrix"

$$Q = \begin{pmatrix} H^0_{\Lambda\Lambda} & I_n\rho/2 \\ I_n\rho/2 & -H^0_{XX} \end{pmatrix}$$

to be positive definite and showed (in Brock and Scheinkman 1977) that Rockefaller's condition implies their condition.

The saddle-point property (in the state-costate space) implies the local stability of $\hat{X}$ in the state space.[5] Returning to the present approach, we relate this property to the $n \times n$ Jacobian matrix of $L(\cdot)$ under curvature conditions imposed on the (ordinary) Hamiltonian $H(X, C, \Lambda)$. In particular, let H_{CC} denote the $n \times n$ Hessian matrix of H with respect to C (with $\partial^2 H/\partial c_i \partial c_j$ as its ij element). Noting (3.7), define $\hat{\Lambda} : \mathbf{X} \mapsto \mathbb{R}^n$ by[6]

$$\hat{\Lambda}(X) = -J_C^G(X, M(X))'^{-1} f_C(X, M(X)). \tag{4.4}$$

Assumption 2. *$H_{CC}(X, C, \Lambda)$ is negative definite in some neighborhood of $(\hat{X}, M(\hat{X}), \hat{\Lambda}(\hat{X}))$ at all states satisfying $L(\hat{X}) = 0$.*

This assumption ensures that, near a steady state $\hat{X}$, the action $C^0(X, \hat{\Lambda}(X))$, defined by (3.5), is the unique (local) maximizer of H and excludes anomalies such as Skiba points.

We now state a property that allows to narrow the list of optimal steady state candidates (identified by Proposition 1) by ruling out some states that are not locally stable:

Proposition 2. *Suppose that assumptions 1-2 hold. If an unconstrained steady state $\hat{X}$ (where $L(\hat{X}) = 0$) is locally stable then*

$$(-1)^n \det(J_X^L(\hat{X})) > 0.$$

Proof. The saddle-point property requires that $(-1)^n \det\left(J^0(\hat{X}, \hat{\Lambda})\right) > 0$ holds at $(\hat{X}, \hat{\Lambda})$. We show that the determinants of $J^0(\hat{X}, \hat{\Lambda})$ and $J_X^L(\hat{X})$ have the same sign. The $2n \times 2n$ Jacobian matrix J^0 can be expressed in

[5] $\hat{X}$ is locally stable if there exists some $\epsilon > 0$ such that (along the optimal trajectory) $\|X(t_0) - \hat{X}\| < \epsilon$ at some t_0 implies $X(t) \to \hat{X}$.

[6] It is clear, noting (3.7) and (4.4), that if $(\hat{X}, \hat{\Lambda})$ is a steady state, then $\hat{\Lambda} = \hat{\Lambda}(\hat{X})$.

terms of H as (see Wirl 1992, Eq. (21))

$$J^0 = \begin{pmatrix} J_X^G + J_C^G J_X^{C^0} & J_C^G J_\Lambda^{C^0} \\ -H_{XX} - H_{XC}\, J_X^{C^0} & \rho I_n - [J_X^G]' - H_{XC}\, J_\Lambda^{C^0} \end{pmatrix} \tag{4.5}$$

where $J_X^{C^0}$ and $J_\Lambda^{C^0}$ are the $n \times n$ Jacobian matrices of $C^0(X, \Lambda)$ with respect to X and Λ, respectively and all functions are evaluated at $(X, C^0(X, \Lambda), \Lambda)$. Differentiating (3.5) with respect to X and Λ gives

$$J_X^{C^0} = -[H_{CC}]^{-1} H_{CX} \tag{4.6}$$

and

$$J_\Lambda^{C^0} = -[H_{CC}]^{-1} H_{C\Lambda} = -[H_{CC}]^{-1} [J_C^G]'. \tag{4.7}$$

(Assumption 2 ensures that $J_X^{C^0}$ and $J_\Lambda^{C^0}$ are well defined.) Substituting in (4.5) yields

$$J^0 = \begin{pmatrix} J_X^G - J_C^G [H_{CC}]^{-1} H_{CX} & -J_C^G [H_{CC}]^{-1} [J_C^G]' \\ -H_{XX} + H_{XC} [H_{CC}]^{-1} H_{CX} & \rho I_n - [J_X^G]' + H_{XC} [H_{CC}]^{-1} [J_C^G]' \end{pmatrix} \tag{4.8}$$

To obtain the Jacobian of L, differentiate (3.3), evaluate all functions at X, $C = M(X)$ and $\Lambda = \hat{\Lambda}(X)$ and use (3.1) to find (after some algebraic manipulations)

$$J_X^L = H_{XX} - H_{XC} [J_C^G]^{-1} J_X^G + \left(\rho I_n - [J_X^G]'\right) [J_C^G]'^{-1} \left(H_{CX} - H_{CC} [J_C^G]^{-1} J_X^G\right). \tag{4.9}$$

To compare $\det(J^0)$ and $\det(J_X^L)$ we use a method to compute the determinant of a $2n \times 2n$ matrix in terms of its four $n \times n$ sub-matrices.[7] Let p, q, r and s be the $n \times n$ matrices in $J^0 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ of (4.8). Let χ be an arbitrary

[7] We are grateful to Shaul Zemel for suggesting this reduction method to us.

$n \times n$ matrix, $\mathbf{0}$ the $n \times n$ matrix of zeros and $\mathcal{X} \equiv \begin{pmatrix} I_n & \mathbf{0} \\ \chi & I_n \end{pmatrix}$. Obviously, $\det(\mathcal{X}) = 1$ hence

$$\det(J^0) = \det(J^0\mathcal{X}) = \det \begin{pmatrix} p + q\chi & q \\ r + s\chi & s \end{pmatrix}.$$

Since $q = -J_C^G[H_{CC}]^{-1}[J_C^G]'$ is nonsingular (Assumptions 1-2), we can choose $\chi = -q^{-1}p$, hence $p + q\chi = \mathbf{0}$ and

$$\det(J^0) = \det(J^0\mathcal{X}) = \det \begin{pmatrix} \mathbf{0} & q \\ r - sq^{-1}p & s \end{pmatrix} = \det(q)\ \det\left(-[r - sq^{-1}p]\right). \tag{4.10}$$

Now,

$$\begin{aligned} r - \ sq^{-1}p &= -H_{XX} + H_{XC}[H_{CC}]^{-1}H_{CX} + \\ &\left(\rho I_n - [J_X^G]' + H_{XC}[H_{CC}]^{-1}[J_C^G]'\right)[J_C^G]'^{-1}H_{CC}[J_C^G]^{-1}\left(J_X^G - J_C^G[H_{CC}]^{-1}H_{CX}\right) \\ &= -J_X^L\,, \end{aligned}$$

where the last equality follows (again, after some algebraic manipulations) by setting $C = M(X)$ in the arguments of the various functions and comparing with (4.9). It follows that

$$\begin{aligned} \det(J^0) &= \det(J_C^G)\det(-[H_{CC}]^{-1})\det([J_C^G]')\det(J_X^L) \\ &= \left(\det(J_C^G)\right)^2\det(-[H_{CC}]^{-1})\det(J_X^L). \end{aligned} \tag{4.11}$$

According to Assumption 2, the matrix $-H_{CC}$ is positive definite, hence (4.11) implies that $\det(J^0)$ and $\det(J_X^L)$ have the same sign.[8] □

Proposition 1 identifies candidates for an optimal steady state and Proposition 2 provides useful information regarding the local stability of each candidate. Together, the two propositions often narrow the list of optimal steady state candidates to a singleton. The next section demonstrates the application of the Propositions in a number of two-state models.

[8]In single-state models ($n = 1$), equation (4.11) reduces to equation (C.10) of Tsur and Zemel (2014).

5 Two-state models

We apply the L-method to a number of two-state models. We begin with three stylized examples that display a range of possible steady states: a unique and stable root; a continuum of unstable roots; and multiple roots of which only one is stable. We then consider the problem of managing hydrologically coupled aquifers.

5.1 Three stylized examples

The three examples below illustrate how the application of the method gives rise to a range of possible steady states.

5.1.1 Example 1: A unique root

Consider the problem

$$v^1(x_1(0), x_2(0)) = \max_{\{c_1,c_2\}} \int_0^\infty [(c_1c_2)^\alpha - x_1 - x_2] \exp(-\rho t)dt, \tag{5.1a}$$

subject to

$$\dot{x}_1 = \gamma c_1 - \delta x_1, \tag{5.1b}$$

$$\dot{x}_2 = \gamma c_2 - \delta x_2 \tag{5.1c}$$

given $x_1(0), x_2(0)$, where $\alpha < 1/2$ is imposed in order to satisfy Assumption 2. The interaction between the states enters via the $(c_1c_2)^\alpha$ term in the objective. Here, $f(X, C) = (c_1c_2)^\alpha - x_1 - x_2$ and $G(X, C) = \gamma C - \delta X$ hence the steady state policy is $M(X) = \frac{\delta}{\gamma}X$, from which we find $J_C^G = \gamma I$, $J_X^G = -\delta I$, $f_X = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $f_C = \alpha(\delta/\gamma)^{2\alpha-1}(x_1x_2)^{\alpha-1} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$.

Using these expressions, (3.3) reduces to

$$L(X) = \frac{\alpha(\rho+\delta)}{\gamma}(\delta/\gamma)^{2\alpha-1}(x_1x_2)^{\alpha-1} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.2}$$

Equation (5.2) admits a unique root

$$\hat{X} \equiv \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \frac{\gamma}{\delta} \left[\frac{\alpha(\rho+\delta)}{\gamma} \right]^{1/(1-2\alpha)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.3}$$

Since no bounds are imposed on X, $\hat{X}$ is the unique candidate for a steady state.

Next we evaluate the Jacobian matrix of L, using (5.2),

$$J_X^L(\hat{X}) = \frac{1}{\hat{x}} \begin{pmatrix} \alpha - 1 & \alpha \\ \alpha & \alpha - 1 \end{pmatrix},$$

where $\hat{x} = \hat{x}_1 = \hat{x}_2 = \frac{\gamma}{\delta} \left[\frac{\alpha(\rho+\delta)}{\gamma} \right]^{1/(1-2\alpha)}$. Thus, $\det(J_X^L) = (1-2\alpha)/\hat{x}^2 > 0$ (since $\alpha < 1/2$) and the stability condition of Proposition 2 is met.

By way of comparison, the 4×4 Jacobian matrix of the modified Hamiltonian system in this example is, using (4.8),

$$J^0 = \begin{pmatrix} -\delta I & -\gamma^2 [H_{CC}]^{-1} \\ \mathbf{0} & (\rho+\delta) I \end{pmatrix}.$$

Thus, $\det(J^0) = [\delta(\rho+\delta)]^2 > 0$, which agrees with the sign of $\det(J_X^L)$. Note that the result regarding the sign of $\det(J^0)$ is independent of the condition that $\alpha < 1/2$. This is a reminder that $\det(J^0) > 0$ is merely a necessary condition for saddle-point stability. Indeed, when $\alpha > 1/2$ the matrix H_{CC} is indefinite (violating Assumption 2) and $\hat{C} = \frac{\delta}{\gamma}\hat{X}$ does not necessarily represent a maximum for the Hamiltonian.

5.1.2 Example 2: A continuum of roots

Consider the problem

$$v^2(x_1(0), x_2(0)) = \max_{\{c_1, c_2\}} \int_0^\infty [c_1^\alpha + c_2^\alpha - x_1 - x_2] \exp(-\rho t) dt, \tag{5.4}$$

subject to

$$\dot{x}_1 = \gamma c_1 - \delta(x_1 + x_2)$$

$$\dot{x}_2 = \gamma c_2 - \delta(x_1 + x_2).$$

given $x_1(0), x_2(0)$, where $\alpha < 1$. The presence of each stock increases the decay rate of the other.

In this example, $M(X) = \frac{\delta}{\gamma}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X$, $J_C^G = \gamma I$, $J_X^G = -\delta \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $f_X = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $f_C = \alpha[\frac{\delta}{\gamma}(x_1 + x_2)]^{\alpha-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f_{XX} = \mathbf{0}$, $f_{CX} = \mathbf{0}$ and $f_{CC} = \alpha(\alpha - 1)\left[\frac{\delta}{\gamma}(x_1 + x_2)\right]^{\alpha-2} I$. Substituting these expressions in (3.3) gives

$$L(X) = \frac{\alpha(\rho + 2\delta)}{\gamma}\left[\frac{\delta}{\gamma}(x_1 + x_2)\right]^{\alpha-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.5}$$

We see that any state vector $\hat{X} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}$ satisfying

$$\hat{x}_1 + \hat{x}_2 = R \equiv \frac{\gamma}{\delta}\left[\frac{\alpha(\rho + 2\delta)}{\gamma}\right]^{1/(1-\alpha)}$$

is a root of L, thus qualifies as a candidate for an optimal steady state.

Differentiating (5.5) with respect to X at $X = \hat{X}$ gives

$$J_X^L(\hat{X}) = \frac{\alpha - 1}{R}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since $\det(J_X^L) = 0$, none of the steady state candidates (the continuum of the roots of L) meets the conditions of Proposition 2 for saddle-point stability.

Indeed, solving for the optimal state process $X^*(t)$, it is found that

$$X^*(t) = \frac{1}{2}\left\{R\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} X(0) - \left[R\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X(0)\right] \mathrm{e}^{-2\delta t}\right\}$$

hence the steady state

$$\hat{X} = \frac{1}{2}\left\{R\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} X(0)\right\}$$

varies continuously with the initial state $X(0)$, or more precisely, with the initial state difference $x_1(0) - x_2(0)$.

Comparing with the standard approach, the 4×4 Jacobian of the modified Hamiltonian system in this example is, using (4.8),

$$J^0 = \begin{pmatrix} J_X^G & -\gamma^2 [f_{CC}]^{-1} \\ \mathbf{0} & \rho I_2 - [J_X^G]' \end{pmatrix}.$$

Thus, $\det(J_X^G) = 0$ implies $\det(J^0) = 0$, in agreement with $\det(J_X^L) = 0$.

5.1.3 Example 3: Unstable roots

Consider the problem

$$v^3(x_1(0), x_2(0)) = \max_{\{c_1, c_2\}} \int_0^\infty [2(x_1 c_1)^{1/2} + 2c_2^{1/2} - w(c_1 + c_2)] \exp(-\rho t) dt, \quad (5.6)$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1) - 2(x_1 c_1)^{1/2} \\ \dot{x}_2 &= c_2 - \delta x_2. \end{aligned}$$

given $X(0) = (x_1(0), x_2(0))'$. For example, x_1 can represent a fish population with a logistic growth, harvested at the rate $2(x_1 c_1)^{1/2}$ (where c_1 is the fishing effort) and the harvest is sold at a fixed unit price. The unit cost of fishing effort is $w > 0$ and $1 > \rho > 0$.[9] The dynamics of x_2 is trivial and is independent of x_1.

In this example,

$$M(X) = \begin{pmatrix} x_1(1 - x_1)^2/4 \\ \delta x_2 \end{pmatrix}, \quad J_C^G = \begin{pmatrix} -2/(1 - x_1) & 0 \\ 0 & 1 \end{pmatrix},$$

$$J_X^G = \begin{pmatrix} (1 - 3x_1)/2 & 0 \\ 0 & -\delta \end{pmatrix}, \quad f_X = \begin{pmatrix} (1 - x_1)/2 \\ 0 \end{pmatrix}$$

and

$$f_C = \begin{pmatrix} 2/(1 - x_1) - w \\ (\delta x_2)^{-1/2} - w \end{pmatrix}.$$

[9] See Example 9.5.1 in Leonard and Long (1992).

Substituting these expressions in (3.3) gives

$$L(X) = \begin{pmatrix} p(x_1) \\ (\rho+\delta)[(\delta x_2)^{-1/2} - w] \end{pmatrix}, \tag{5.7}$$

where $p(x)$ is the quadratic polynomial

$$p(x) = \frac{1}{4}\left[-3wx^2 + (4w - 8 - 2w\rho)x + 4(1-\rho) - w(1-2\rho)\right],$$

which has two real roots, denoted $\hat{x}_1^{\pm}$, where $\hat{x}_1^+ > \hat{x}_1^-$ and $1 > \hat{x}_1^+ > 0$.[10] The function L, then, admits two roots: $\hat{X}^{\pm} = \begin{pmatrix} \hat{x}_1^{\pm} \\ \hat{x}_2 \end{pmatrix}$, where $\hat{x}_2 = 1/(\delta w^2)$.

To study the stability of each root, we consider the Jacobian matrix

$$J_X^L(\hat{X}^{\pm}) = \begin{pmatrix} \frac{dp(\hat{x}_1^{\pm})}{dx} & 0 \\ 0 & -\delta(\rho+\delta)w^3/2 \end{pmatrix}.$$

The coefficient $-3w/4$ of the quadratic term of $p(x)$ is negative hence dp/dx is positive at the smaller root $\hat{x}_1^-$ and $\det[J_X^L(\hat{X}^-)] < 0$, violating the stability condition. In contrast, the curve of p decreases at $\hat{x}_1^+$ hence $\det[J_X^L(\hat{X}^+)] > 0$ since both eigenvalues are negative, hence this feasible root qualifies as a stable steady state.

5.2 The management of coupled aquifers

The recharge of many groundwater sources is derived, in addition to precipitation, also from lateral flows between adjacent aquifers or between cells within an aquifer. As a result, groundwater management often requires the consideration of multiple aquifers or multiple cells within an aquifer (see examples in Zeitouni and Dinar 1997, Athanassoglou et al. 2012). We apply the L-method to the case of two adjacent aquifers. The extension to $n > 2$ is straightforward.

[10] Write the discriminant of $p(x)$ as $\Delta^2 = [w(\rho+1)-4]^2/4 + w(\rho+1) > 0$ to verify that the roots are real. Moreover, $p(1) = -(\rho+1) < 0$ while $\frac{dp}{dx}(1) = -[w(\rho+1)+4]/2 < 0$ hence both roots fall short of unity. On the other hand, $p((1-\rho)/2) = w(1+\rho)^2/16 > 0$ hence $\hat{x}_1^+ > (1-\rho)/2 > 0$ as required for a feasible state.

Water is extracted from two adjacent aquifers, whose stocks $X = (x_1, x_2)'$ evolve in time according to

$$\dot{X} = G(X, C) = \begin{pmatrix} r_1 - c_1 - \delta(x_1 - x_2) \\ r_2 - c_2 - \delta(x_2 - x_1) \end{pmatrix}, \tag{5.8}$$

where the vector $C = (c_1, c_2)'$ represents the extraction rates, r_1 and r_2 are the (constant) exogenous recharge rates, and δ is a flow rate describing the lateral flow between the two aquifers when their stocks differ. Extraction at the rates $C = (c_1, c_2)'$ generates the instantaneous benefit

$$f(X, C) = \alpha(c_1 + c_2) - (c_1^2 + c_2^2)/2 - (z_0 - z_1 x_1)c_1 - (z_0 - z_2 x_2)c_2 \tag{5.9}$$

where $\alpha > 0$ is the maximal marginal benefit from the extraction of either stock, and z_0, z_1, z_2 are extraction costs parameters. A water policy $\{C(t),\, t \geq 0\}$ is feasible if the ensuing stocks $X(t) \in \mathbf{X} \equiv [0, \bar{x}_1] \times [0, \bar{x}_2]$, where $\bar{x}_1 \leq z_0/z_1$ and $\bar{x}_2 \leq z_0/z_2$ are the aquifers capacity bounds. These bounds ensure that the unit extraction costs (which decrease when the stocks are larger) never turn negative. The optimal policy is the feasible policy that maximizes

$$\int_0^\infty f(X(t), C(t)) e^{-\rho t} dt$$

subject to (5.8) given $X(0)$, where ρ is the time rate of discount.

In this example,

$$M(X) = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} - \delta \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} X, \qquad J_C^G = -I,$$

$$f_X = \begin{pmatrix} z_1 r_1 \\ z_2 r_2 \end{pmatrix} - \delta \begin{pmatrix} z_1 & -z_1 \\ -z_2 & z_2 \end{pmatrix} X, \quad J_X^G = -\delta \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and

$$f_C = \begin{pmatrix} \alpha - z_0 - r_1 \\ \alpha - z_0 - r_2 \end{pmatrix} + \begin{pmatrix} z_1 + \delta & -\delta \\ -\delta & z_2 + \delta \end{pmatrix} X.$$

Substituting these expressions in (3.3) gives

$$L(X) = J_X^L X + L_0 \tag{5.10}$$

where

$$J_X^L = \begin{pmatrix} -(2\delta+\rho)(z_1+\delta) & \delta(z_1+z_2+2\delta+\rho) \\ \delta(z_1+z_2+2\delta+\rho) & -(2\delta+\rho)(z_2+\delta) \end{pmatrix} \tag{5.11}$$

is the Jacobian of $L(X)$ and

$$L_0 = \begin{pmatrix} \delta(r_1-r_2)+r_1z_1+\rho(z_0-\alpha+r_1) \\ \delta(r_2-r_1)+r_2z_2+\rho(z_0-\alpha+r_2) \end{pmatrix}.$$

When J_X^L is nonsingular, $L(X)$ admits a unique root

$$\hat{X} = -[J_X^L]^{-1}L_0. \tag{5.12}$$

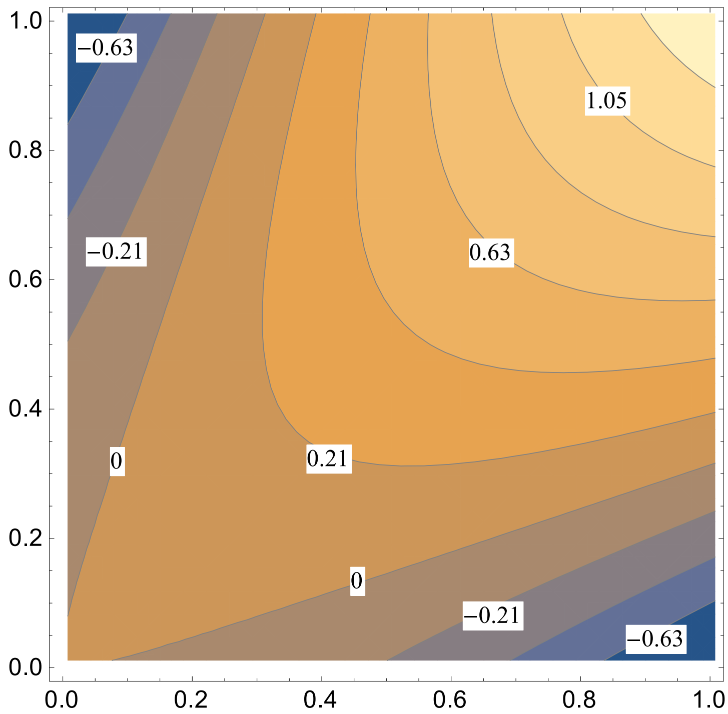

Figure 1: Contours of $100\times\det(J_X^L)$ plotted in the z_1 (horizontal) - z_2 (vertical) plane. The $\det(J_X^L)=0$ contours consist of two curves arranged symmetrically on both sides of the $z_2=z_1$ diagonal. The determinant is positive inside the region confined between the $\det(J_X^L)=0$ contours and is negative outside it.

Notice that J_X^L depends on ρ (the discount rate), δ (the lateral flow parameter), and z_1 and z_2 (the marginal extraction costs). Given values of ρ and δ (e.g., the values listed in Table 1), we consider $\det(J_X^L)$ as a function of the marginal extraction costs (z_1, z_2). Figure 1 depicts the contours of constant values of $\det(J_X^L)$, which indicate the regions in the (z_1, z_2) plane at which $\det(J_X^L)$ is positive or negative. These regions are separated by the two curves along which $\det(J_X^L) = 0$, denoted $z_2^{\pm}(\cdot)$ and determined from (5.11) as

$$\begin{aligned} z_2^{\pm}(z_1) = \frac{1}{2\delta^2} \left[2\delta^2\rho + \delta\rho^2 + z_1(2\delta^2 + 4\delta\rho + \rho^2)\right] \\ \pm \frac{\sqrt{\rho}(2\delta+\rho)}{2\delta^2}\sqrt{\delta^2\rho + z_1^2(4\delta+\rho) + \delta z_1(4\delta + 2\rho)} \end{aligned}$$

which implies $z_2^-(z_1) < z_1 < z_2^+(z_1)$. Along these curves, the solution (5.12) is not well-defined and no internal steady state exists (except for the special case in which the vector L_0 is parallel to both columns of J_X^L and $L(\cdot)$ yields a continuum of unstable roots, a situation akin to that obtained in Example 5.1.2). Setting $z_1 = z_2$ in (5.11), we find that $\det(J_X^L) > 0$ along the diagonal of Figure 1, hence the determinant is positive in the region confined between the $z_2^{\pm}(\cdot)$ curves. It follows that the vector $\hat{X}$ of (5.12) obtained with a (z_1, z_2) pair in this positive region meets the stability condition of Proposition 2.

Table 1: Parameter values.

Parameter	Value	Description
ρ	0.03	discount rate
δ	0.1	lateral flow rate
z_0	10	maximal unit extraction cost
α	10	maximal marginal benefit
r_1	1	recharge rate of aquifer 1
r_2	1	recharge rate of aquifer 2

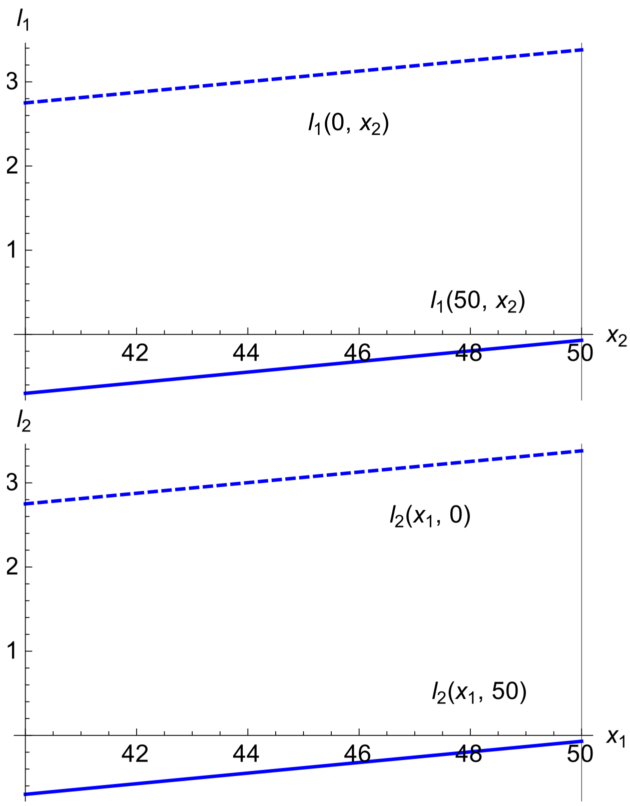

Figure 2: L-function components in the case $z_1 = z_2 = 0.2$ and $\bar{x}_1 = \bar{x}_2 = 50$. Upper panel: $l_1(50, x_2)$ and $l_1(0, x_2)$ for $x_2 \in [0, 50]$. Lower panel: $l_2(x_1, 50)$ and $l_2(x_1, 0)$ for $x_1 \in [0, 50]$. In all cases, the signs of $l_i(\cdot, \cdot)$ violate the condition of Proposition 1 for a corner steady state.

As an example, we consider $z_1 = z_2 = 0.2$ and note, observing Figure 1, that $\det(J_X^L) > 0$. Now, $\bar{x}_i = z_0/z_i = 50$, $i = 1, 2$, while equation (5.12) gives $\hat{X} = (38.33,\ 38.33)' \in \mathbf{X} = [0, 50] \times [0, 50]$, hence the root is feasible. To check if an optimal steady state can fall on a boundary (i.e., $\hat{x}_i = 0$ or 50, $i = 1, 2$) we see in Figure 2 that $l_1(50, x_2) < 0$ and $l_1(0, x_2) > 0$ for all $x_2 \in [0, 50]$; likewise, $l_2(x_1, 50) < 0$ and $l_2(x_1, 0) > 0$ for all $x_1 \in [0, 50]$. It follows that any corner state violates the conditions of Proposition 1, hence does not qualify as an optimal steady state. We conclude that $\hat{X} = (38.33, 38.33)'$ is the only candidate for a stable steady state.

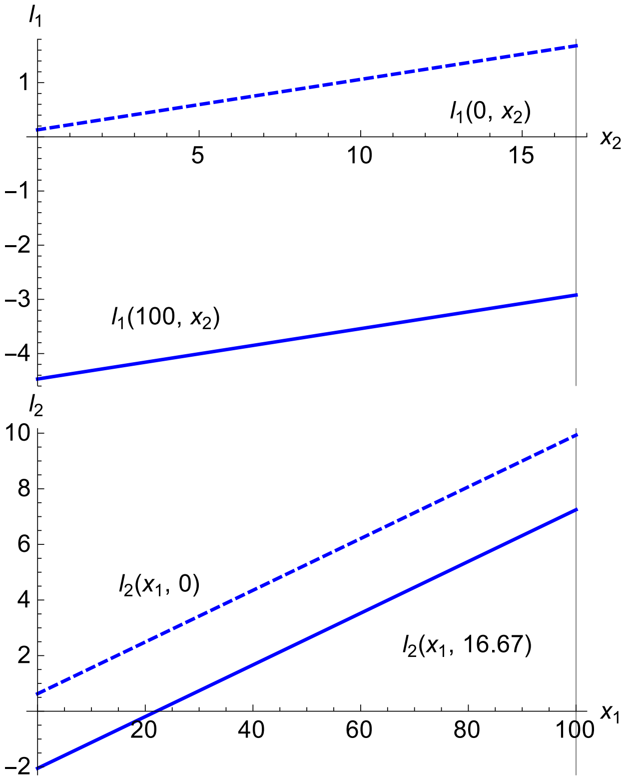

Figure 3: L-function components in the case $z_1 = 0.1$ and $z_2 = 0.6$, corresponding to $\bar{x}_1 = 100$ and $\bar{x}_2 = 16.67$. Upper panel: $l_1(100, x_2)$ and $l_1(0, x_2)$ for $x_2 \in [0, 16.67]$. Lower panel: $l_2(x_1, 16.67)$ and $l_2(x_1, 0)$ for $x_1 \in [0, 100]$. Only the $l_2(x_1, 16.67)$ line can obtain the correct sign for a corner steady state.

Consider now the case $z_1 = 0.1$ and $z_2 = 0.6$, where the effect of the stock on the extraction cost is considerably larger in the second aquifer. Figure 1 reveals that $\det(J_X^L) < 0$ in this case, implying that there exists no *unconstrained* steady state that is locally stable. Thus, only *constrained* steady states (that fall on the boundaries: $\hat{x}_i = 0$ or $\bar{x}_i$, $i = 1, 2$) may qualify. In this case, the upper bounds are $\bar{x}_1 = z_0/z_1 = 100$ and $\bar{x}_2 = z_0/z_2 = 16.67$. Observing Figure 3, we see (upper panel) that $l_1(100, x_2) < 0$ and $l_1(0, x_2) > 0$ for all $x_2 \in [0, 16.67]$, ruling out (by virtue of Proposition 1) the possibility of a constrained steady state with either $\hat{x}_1 = 100$ or $\hat{x}_1 = 0$. Likewise, the lower panel of Figure 3 shows that $l_2(x_1, 0) > 0$ for all $x_1 \in [0, 100]$, ruling out the possibility of a constrained steady state with $\hat{x}_2 = 0$. In contrast, the $l_2(x_1, 16.67)$ line obtains the correct (positive) sign for an optimal corner state at all $x_1 > 22$. Since the constrained steady state must have $\hat{x}_1$ internal, the choice of $\hat{x}_1$ must satisfy $l_1(\hat{x}_1, 16.67) = 0$ (Proposition 1) giving $\hat{x}_1 = 36.52$. Thus, the corner state $\hat{X} = (36.52, 16.67)'$ is the unique candidate for an optimal steady state. We see that, in this example, Propositions 1 and 2 narrow the list of optimal steady state candidates to a single state.

6 Concluding comments

The L-method has originally been proposed in Tsur and Zemel (2001, 2014) to study long-run behavior of single-state models. The present work extends the method to multi states models. In particular, unconstrained steady states must be roots of $L(\cdot)$ while at corner steady states the relevant component of L must obtain the correct sign. Similarly, the result that the one-dimensional L must decrease at a locally stable steady state translates to a corresponding condition on the sign of the determinant of the Jacobian matrix of the n-dimensional L-function, because this matrix is closely related to the Jacobian matrix of the $2n$-dimensional dynamical system that governs the evolution in time of the optimal state-costate processes.

The L-function is readily obtained from the model's primitives, and its application depends on minimal conditions – essentially the conditions needed for the existence of an optimal policy. These properties open the way to study long-run behavior in a large class of multi-state dynamic systems.

References

Athanassoglou, S., Sheriff, G., Siegfried, T. and Huh, W.: 2012, Optimal mechanisms for heterogeneous multi-cell aquifers, *Environmental and Resource Economics* **52**(2), 265–291.

Benhabib, J. and Nishimura, K.: 1979, The hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth, *Journal of Economic Theory* **21**(3), 421 – 444.

Brock, W. A. and Scheinkman, J. A.: 1976, Global asymptotic stability of optimal control systems with applications to the theory of economic growth, *Journal of Economic Theory* **12**(1), 164 – 190.

Brock, W. A. and Scheinkman, J. A.: 1977, *The global Asymptotic Stability of optimal control with applications to dynamic economic theory*, North Holand, Amsterdam, chapter 8, pp. 173–205.

Cass, D. and Shell, K.: 1976, Introduction to hamiltonian dynamics in economics, *Journal of Economic Theory* **12**(1), 1 – 10.

Dasgupta, P. and Mäler, K.-G.: 2003, The economics of non-convex ecosystems: Introduction, *Environmental & Resource Economics* **26**, 499–525.

Dockner, E. J. and Feichtinger, G.: 1991, On the optimality of limit cycles in dynamic economic systems, *Journal of Economics* **53**(1), 31–50.

Feichtinger, G., Novak, A. and Wirl, F.: 1994, Limit cycles in intertemporal adjustment models: Theory and applications, *Journal of Economic Dynamics and Control* **18**(2), 353 – 380.

Kurz, M.: 1968, The general instability of a class of competitive growth processes, *The Review of Economic Studies* **35**(2), 155–174.

Leonard, D. and Long, N.: 1992, *Optimal Control Theory and Static Optimization in Economics*, Cambridge University Press.

Levhari, D. and Liviatan, N.: 1972, On stability in the saddle-point sense, *Journal of Economic Theory* **4**(1), 88 – 93.

Rockafellar, R. T.: 1976, Saddle points of hamiltonian systems in convex lagrange problems having a nonzero discount rate, *Journal of Economic Theory* **12**(1), 71 – 113.

Sorger, G.: 1989, On the optimality and stability of competitive paths in continuous time growth models, *Journal of Economic Theory* **48**(2), 526 – 547.

Tsur, Y. and Zemel, A.: 2001, The infinite horizon dynamic optimization problem revisited: A simple method to determine equilibrium states, *European Journal of Operational Research* **131**, 482–490.

Tsur, Y. and Zemel, A.: 2014, Steady-state properties in a class of dynamic models, *Journal of Economic Dynamics and Control* **39**, 165 – 177.

Wirl, F.: 1992, Cyclical strategies in two-dimensional optimal control models: Necessary conditions and existence, *Annals of Operations Research* **37**(1), 345–356.

Wirl, F.: 1995, The cyclical exploitation of renewable resource stocks may be optimal, *Journal of Environmental Economics and Management* **29**(2), 252 – 261.

Zeitouni, N. and Dinar, A.: 1997, Mitigating negative water quality and quality externalities by joint management of adjacent aquifiers, *Environmental and Resource Economics* **9**(1), 1–20.