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A Programming Model for Optimal Patterns of Investment, Production and Consumption Over Time*

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ABSTRACT

This paper presents a formulation of an applicable linear programming model for a growing economic unit. The model covers all aspects of activity of such a unit: production, investment, consumption and credit operations. Alternative consumption functions are incorporated. Particular care is taken to make the suggested system consistent with economic theory, and it is shown that correct formulation will yield solutions which maintain temporal equilibrium throughout. Diversions from the correct formulation are also analyzed.

This work is an attempt to construct a programming model of optimal resource allocation in an economic unit over time. Particular care will be taken to make the model practically applicable as well as theoretically sound. There is no need to dwell on the importance of the problem of dynamic resource allocation. It is relevant to the understanding of the behaviour of the household, to the budgeting of the firm and to planning of economic development. We hope that this paper will be a contribution toward better solutions of problems in these areas.

Our point of departure will be Hirshleifer's "On the Theory of Optimal Investment Decision" (1958), where a model first suggested by Fisher is extended. This work is well known and it will not be reviewed here. It will suffice to remind the reader that Hirshleifer found that investment decisions should always be made simultaneously with consumption decisions of the economic unit. Hirshleifer's model is a theoretical analysis employing indifference curves and continuous transformation curves. We shall present a translation of his analysis into a mathematical programming model. Such a formulation has already been suggested by Baumol and Quandt (1965), whose model includes the businessman's welfare function in the objective function of the linear programming problem. They made Hirshleifer's case into an applicable programming model only in a very limited sense,** since welfare functions are not observable, and we doubt that many businessmen can formulate their own function.

We shall try to introduce observable consumption functions into the programming model, discuss the difficulties that arise and suggest practical solutions. Our model will be more complete than Baumol and Quandt's in covering financial and current production aspects of the economic unit, as well as investment activities.

The Appendix presents a three-year simplex tableau of what we call Probelm IV which should make it easier for the reader to follow the mathematical formulations of the models.

A PROGRAMMING FORMULATION OF HIRSHLEIFER'S CASE

We start by introducing some notations which will also be used in subsequent sections. We adopt the notational convention of using capital letters for matrices only. Lower case letters are vectors, unless indicated otherwise. Greek letters denote scalars.

Let

x_t^p = level of current production activities at t, $t = 1, 2, \dots, T$;

x_t^i = level of "real" investment activities started at t;

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** Ophir suggested a capital accumulation model of similar nature, but did not consider consumption as an endogenous variable.

ϕ_t^f = amount of funds lent ("financial investment") at t;
ϕ_t^b = amount of funds borrowed at t;
y_t = income* at t, $t = 1, 2, \ldots, T-1$;
c_t = consumption at end of year t, $t = 1, 2, \ldots, T-1$;
β_t^f = lending discount rate at t;
β_t^b = borrowing discount rate at t;
ω = wealth at the horizon T;
A_t^p = matrix of input coefficients related to limited factors per unit of current production activities at t (technology matrix);
$A_{\tau t}^i$ = the technology matrix describing input requirements at t per unit of investment activities started at $\tau \leqq t$;
$B_{\tau t}^i$ = a matrix describing outputs of investment activities in the same fashion as $A_{\tau t}^i$ describes inputs;**
k_t^p = pecuniary requirements per unit of current production activities at t;
$k_{\tau t}^i$ = pecuniary requirements at t per unit of investment activities started at $\tau \leqq t$;
μ_t = amount of funds available exogenously at t;
q_1 = available quantities of production factors at $t = 1$;
q_t = the non-obsolete portion of q_1 at t ($t = 2, 3, \ldots, T$);
r_τ^i = residual value at the end of T per unit of investment activities started at $\tau \leqq T$;
Ω = value of q_T at the end of T;
r_t^p = revenue per unit level of current production activities at t.

Thus an investment project, in the present model, does not directly contribute income but adds to the stock of reproducable assets of the economy (see Appendix). It should be noted that we can take care of depreciation by adjusting the elements of q_t and $B_{\tau t}^i$. Anticipated technological changes are indicated by the fact that the technology matrices are subscripted by t.

We do not have to assume constant interest rates. Declining lending or, what is more often the case, rising borrowing rates can be introduced via continuous or step functions without changing qualitatively any of the subsequent results.***

Problem I (Hirshleifer's case) is to maximize

$$F(c_1, c_2, \ldots, c_{T-1}, \omega) \quad (1)$$

subject to

$$-r_{t-1}^p x_{t-1}^p - \beta_{t-1}^f \phi_{t-1}^f + \beta_{t-1}^b \phi_{t-1}^b + c_{t-1} + k_t^p x_t^p + \textstyle\sum_\tau k_{\tau t}^i x_\tau^i + \phi_t^f - \phi_t^b \leqq \mu_t \quad (2)$$

$$A_t^p x_t^p + \textstyle\sum_\tau A_{\tau t}^i x_\tau^i - \sum_\tau B_{\tau t}^i x_\tau^i \leqq q_t \quad (3)$$

$$-r_T^p x_T^p - \beta_T^f \phi_T^f + \beta_T^b \phi_T^b - \textstyle\sum_\tau r_\tau^i x_\tau^i + \omega \leqq \Omega \quad (4)$$

$$t = 1, 2, \ldots, T$$

and non-negativity of all variables, where F is the time preference function.

Some remarks, which will apply to subsequent models as well, are in order. First, for $t = 1$, only the last four terms of (2) are relevant. Secondly, from the formulation it seems as if we assume all credit to be on an annual basis. This is done only for notational convenience. However, even if such an assumption were really necessary it would merely imply a "perfect" finance market in the sense that the borrower is certain of being able to secure whatever amounts he deems profitable in any future year. In general, any short and long run combinations of borrowing or lending options can be introduced as any experienced programmer will recognize.

The present formulation differs from Hirshleifer's in one important aspect. He considers only the transformation curve—the efficiency frontier—and leaves the individual projects that give rise to this curve in the background. This raises difficult questions for longer then two year periods (see Bailey's extension of Hirshleifer's analysis (*Bailey*, 1959)) which are not encountered in this programming model. As usual, the individual activities are independent; that is, projects are not mutually exclusive. Mutual exclusiveness may be treated by formulating alternative programs and choosing the best (compare to *Hirshleifer*, 1958, fig. 5).†

* Both y_t and c_t are, of course, scalars.
** The assumption that inputs and outputs of investment activities can be described by disjoint matrices is made to avoid more notational complications.
*** For example see Yaron and Heady (1961), Plessner and Heady (1965).
† See also Weingartner (1966), who discusses extensively programming of mutually exclusive investment projects, but assumes an exogenously given discount rate.

Finally, even if the time preference function is given, application involves difficult problems. Future prices and technology, and in particular residual values of assets (r_τ^i), have to be assessed. This, however, is common to all long run practical models; and, in fact, every businessman making an important decision is, explicitly or implicitly, predicting future economic magnitudes.

Since the internal rate of return plays a pivotal role in the subsequent discussion, we think it instructive to show how it may be theoretically calculated from the solution to the problem. To this end we associate with (2) the "shadow prices" λ_t^μ, with (3)—the vector v_t, and with (4)—the imputed value η.

Consider the lending activity of year T. We define the internal rate of return, in an obvious way, by

$$1 + \rho_T^f = \beta_T^f .$$

If lending actually takes place, then the dual equation associated with the activity reads

$$\lambda_T^\mu - \beta_T^f \eta = 0$$

from which

$$1 + \rho_T^f = \beta_T^f = \frac{\lambda_T^\mu}{\eta} . \tag{5}$$

Next, consider an investment activity started at the beginning of T. We define

$$1 + \rho_T^i = \frac{r_T^{i\cdot} - (a_{TT}^i v_T/\eta)}{k_{TT}^{i\cdot}} , \tag{6}$$

where $r_T^{i\cdot}$ and $k_{TT}^{i\cdot}$ are elements of r_T^i and k_{TT}^i, respectively, a_{TT}^i being a vector of A_{TT}^i. ($r_T^{i\cdot}$, $k_{TT}^{i\cdot}$ and $a_{TT}^{i\cdot}$ belong, of course, to *the same* activity column. We avoided the identifying column index to simplify notation.)

This definition is, to be sure, a common one. The division of v_T by η is appropriate because the costs of inputs have to be expressed in values of the end of T. While v_T is in terms of dollars at the horizon, η is the value to the economic unit of a dollar at the end of T.

If the activity under consideration is operated, the relevant dual equation reads—

$$k_{TT}^{i\cdot} \lambda_T + a_{TT}^i v_T - r_T^{i\cdot} \eta = 0$$

from which, taken together with (6),

$$1 + \rho_T^i = \frac{\lambda_T^\mu}{\eta} . \tag{7}$$

A comparison of (5) and (7) reveals

$$1 + \rho_T^i = 1 + \rho_T^f .$$

In general, it is easy to check that the internal rate of return will be the same for every investment project which is undertaken and will be greater than or equal to the "going" (market) interest rate in lending opportunities.

An internal interest rate is also implicit in every production activity. We define this rate in year t, ρ_t^p, by

$$1 + \rho_t^p = \frac{r_t^{p\cdot} - a_t^p (v_t/\lambda_{t+1}^\mu)}{k_t^{p\cdot}} . \tag{8}$$

Writing the appropriate dual equation, and assuming that production takes place, one finds

$$\frac{\lambda_t^\mu}{\lambda_{t+1}^\mu} = 1 + \rho_t^p . \tag{9}$$

Similarly to (8), we can define $1 + \rho_t^i$ for every t—the internal rate of returns of investment activities—which will also satisfy (9). The equality of the rates of all operated activities is kept, as the reader may check, for every year t. It is, therefore, possible to define a *common* rate of discount, ρ_t^* for the year t,

$$\rho_t^* = \frac{\lambda_t^\mu}{\lambda_{t+1}^\mu} - 1 . \tag{10}$$

Also,

$$\beta_t^b \geqq 1 + \rho_t^* \geqq \beta_t^f , \tag{11}$$

as Hirshleifer has shown. From (7), (9) and (10) we get

$$\lambda_t^\mu = \eta \prod (1 + \rho_\tau^*) , \tag{12}$$

the financial shadow prices are the compound rates of interest multiplied by the value of the dollar at the horizon.

Finally, an immediate result from the dual equations is that the marginal rates of substitution between consumption in any two successive periods is given by

$$-\frac{\partial F/\partial c_t}{\partial F/\partial c_{t+1}} = \frac{dc_{t+1}}{dc_t} = -\frac{\lambda_{t+1}^\mu}{\lambda_{t+2}^\mu} = -(1 + \rho_{t+1}^*) , \tag{13}$$

as one would expect.

PREDETERMINED CONSUMPTION OUTLAYS

As pointed out already, time preference functions are generally unknown. In the simplest alternative model we suggest, consumption is only a function of time and independent of income.

Let c_t^* denote the minimum annual consumption requirement. Then our Problem II is to find non-negative values which maximize—

$$\psi = \omega \tag{14}$$

subject to (2), (3), (4) and

$$c_t \geqq c_t^* . \tag{15}$$

One important characteristic of Problem II is that here, unlike in Problem I, we have at optimum

$$\eta = 1 . \tag{16}$$

This puts us in a situation of having our programming horizon as a definite zero point on the time axis such that one dollar at that point is worth exactly one dollar. In view of (16), we have

$$\lambda_T^\mu = 1 + \rho_T^*$$

and (9), (10), (11) and (12) can be verified to hold.

The major conceptual difference between the two problems arises from the fact that in the latter consumption is treated as a burden on the system, and is not being solved for endogenously. The burden can be forcefully demonstrated if we associate with (15) the dual value λ_t^c and note that

$$\lambda_t^c = \lambda_{t+1}^\mu . \tag{17}$$

It is interesting to point out, that (17) implies

$$-\frac{\partial\psi/c_t^*}{\partial\psi/\partial c_{t+1}^*} = \frac{dc_{t+1}^*}{dc_t^*} = -\frac{\lambda_t^\mu}{\lambda_{t+1}^\mu} = -(1+\rho_{t+1}^*) . \tag{18}$$

Obviously, (18) is not the same as (13). Both, however, are necessary (though not sufficient) conditions for inter-temporal rquilibrium in consumption.

OBSERVABLE CONSUMPTION FUNCTIONS

Problem I, with utility function as the objective function, is not applicable (but see *Baunol and Quandt*, 1965). Problem II formulation, on the other hand, is unsatisfactory. Theory and practice teach that consumption is a function of income and wealth and not just a predetermined outlay, We shall now present two versions of our model with consumption as an endogenous variable. In the first version we assume the Keynsian consumption function,

$$c_t = \alpha_t + \gamma y_t . \tag{19}$$

In the second case we let consumption be a function of the worth of the assets of the programmed economy at the horizon,

$$c_t = \alpha_t' + \kappa\omega . \tag{20}$$

Both (19) and (20) are observable. Similar functions have already been estimated (*Ferber*, 1966). They are suggested here as "proxies" for the unobservable welfare function.

We now turn to incorporate these functions in our model. Starting with (19), note that income is, in our case, a linear combination of the operated activities.

$$\begin{aligned} y_t &= (r_t^p - k_t^p)x_t^p + (\beta_t^f - 1)\phi_t^f - (\beta_t^b - 1)\phi_t^b \\ &\equiv z_t x_t^p + \rho_t^f\phi_t^f - \rho_t^b\phi_t^b . \end{aligned} \tag{21}$$

This makes possible the formulation of Problem III: Maximize (14) subject to (3), (4) and

$$\begin{aligned} &-(r_{t-1}^p - \gamma z_{t-1})x_{t-1}^p - [1 + (1-\gamma)\rho_{t-1}^f]\phi_{t-1}^f + \\ &+ [1 + (1-\gamma)\rho_{t-1}^b]\phi_{t-1}^b + k_t^p x_t^p + \sum k_{\tau t}^i x_\tau^i + \\ &+ \phi_t^f - \phi_t^b \leqq \mu_t - \alpha_{t-1} . \end{aligned} \tag{2'}$$

Consumption appears in Problem III, implicity, as a leakage—in (2′) only the amounts not consumed (gross saving) are carried over to next year. Equation (2′) makes sure that the funds diverted to consumption will be determined by (19).

The formulation of Problem III is quite convenient. It will, however, cause misallocation of resources in the programmed economy. To see this, we turn to the dual.

As in Problem II, $\eta = 1$. Internal rates of return are defined as in (6) and (8). However, writing the dual equation for an operated production activity, we get [λ_t^μ is now the shadow price associated with (2′)]

$$k_t^p \lambda_t^\mu + a_t^p v_t - (r_t^p - \gamma z_t)\lambda_{t+1}^\mu = 0 \tag{22}$$

and

$$\frac{\lambda_t^\mu}{\lambda_{t+1}^\mu} = \frac{r_t^{p\cdot} - (a_t^p v_t/\lambda_{t+1}^\mu)}{k_t^{p\cdot}} - \gamma\frac{z_t}{k_t^{p\cdot}} =$$

$$= 1 + \rho_t^p - \gamma\frac{z_t}{k_t^{p\cdot}} \qquad (23)$$

The value of $1 + \rho_t^p$ is the marginal rate of return of a production activity. The ratio $\lambda_t^\mu/\lambda_{t+1}^\mu$ is the same for all activities. Since the last term of the right hand side of (23) will vary from one activity to another, it is evident that the marginal rate of return of the different production activities will not be equal at the optimum solution to the present problem. It is also possible to show discrepancies between the internal rates of return and the ratios of the shadow prices for the other kinds of activities. They will, however, not appear in investment activities, which do not contribute to current income and have, therefore, no consumption elements in their columns. The economic reason for these findings can easily be explained. In our model consumption is imposed as an income tax. By investing in real assets the programmed unit can avoid the penalty of the tax. However, we are trying to program an economy which regards consumption as "good" not as "bad." Given that consumption contributes to the welfare of the programed economy, resources will be misallocated by the program.

The direction of this misallocation can also be recognized from (23). z_i is the value added in a production activity in year t. The higher the ratio of value added to initial pecuniary requirements (the ratio $z_i/k_t^{p\cdot}$), the larger this discrepancy of the marginal contribution of the respective activity from the overall marginal rate of return of the program. In other words, the solution to Problem III will favor activities with relatively low value added, thus avoiding consumption.

Some comfort can, however, be derived from the fact that business executives of large corportations may still find the formulation of Problem III useful. Assume that we are programming such an enterprise and that c_t is not consumption, but dividends paid in year t, which the management promised stock holders to pay as a function of income. If management does not attach any value to these dividends, it will try to follow the recommendations of a model such as our Problem III.

ELIMINATING MISALLOCATION

Problem III can be amended to eliminate its misallocative nature. Let Problem IV be: Maximize

$$\psi = \sum_t \delta_t c_t + \omega \qquad (24)$$

subject to (2′), (3), (4) and

$$-\gamma z_t x_t^p - \gamma\rho_t^f\phi_t^f + \gamma\rho_t^b\phi_t^b + c_t \leqq \alpha_t\,. \qquad (25)$$

Eq. (25) is the consumption function incorporated in the linear programming formulation. The δ_t are arbitrary positive constants.

There will be no misallocation of resources if the objective function coefficients in (25) are chosen in such a way that in the solution to Problem IV

$$\delta_t = \lambda_{t+1}^\mu \text{ for every } t. \qquad (26)$$

We shall call a solution maintaining (26) *an unbiased solution*. It is our assumption that in the system we program an unbiased solution exists. The assumption is based on the expectation that in the real world an unibased program exists. We shall later show that such a solution can be found.

To see that misallocation has been eliminated, consider the dual equations corresponding to the c_t columns of Problem IV. They are

$$\lambda_t^c = \delta_t\,. \qquad (27)$$

Equality will always be maintained, since $c_t > 0$ for every t. (This point is seen clearly from the simplex tableau in the Appendix. The activities c_t do not require factors of production, but contribute to the objective function.)

The dual equations of the production activities will now contain an element from (25). Instead of (22) we have

$$k_t^{p\cdot}\lambda_t^\mu + a_t^p v_t - (r_t^{p\cdot} - 8z_i)\lambda_{t+1}^\mu + \gamma z_i\lambda_t^c = 0. \quad (22')$$

However, if $\delta_t = \lambda_{t+1}^\mu$ one gets $\lambda_t^c = \lambda_{t+1}^\mu$. Introducing this last equality into (22′) will eliminate altogether the misallocation term $\gamma(z_i/k_t^{p\cdot})$ that appeared in (23).

The economic interpretation is simple. In an unbiased solution consumption dollars are given the same weight in the objective function as dollars carried over to next year's production and investment activities. Thus consumption is no longer a burden. The equality of the transformation rates in production and consumption can be shown here, as in (18). Also, the remark we made at the end of Section B, about the maintenance of the condition for equilibrium, can be repeated more emphatically with respect to Problem IV.

We now show that an unbiased solution (whose existence we assumed) is attainable. Our procedure will simply be to program with alternative δ_t values. We shall presently claim that if we try enough, we may find the set of δ_t values that will maintain the equality of (26). We do not suggest a practical or efficient search method, we just want to show that it is possible to reach an unbiased solution.

Proposition. An unbiased solution consistent with (26) is attainable in a finite number of computational operations.*

To prove the proposition, we start by showing that the λ_t^μ values are restricted to a bounded and *identifiable* region. This is done by inserting (10) into (11) (both equations apply in Problem IV too)

$$\beta_t^b \geqq \lambda_t^\mu/\lambda_{t+1} \geqq \beta_t^f. \tag{28}$$

For $t = T$ (recall, $\eta = 1$)

$$\beta_T^b \geqq \lambda_T^\mu \geqq \beta_T^f. \tag{29}$$

Since the λ_t^μ are compounded interest rates (12), boundedness is not surprising—the annual internal rates are bounded between the lending and the borrowing rate. Reference to financial operations was made for convenience. Zero will always be a lower bound and an upper bound can be found by considering the production or investment activity with the highest returns to the dollar when all shadow prices of production factors are zero.

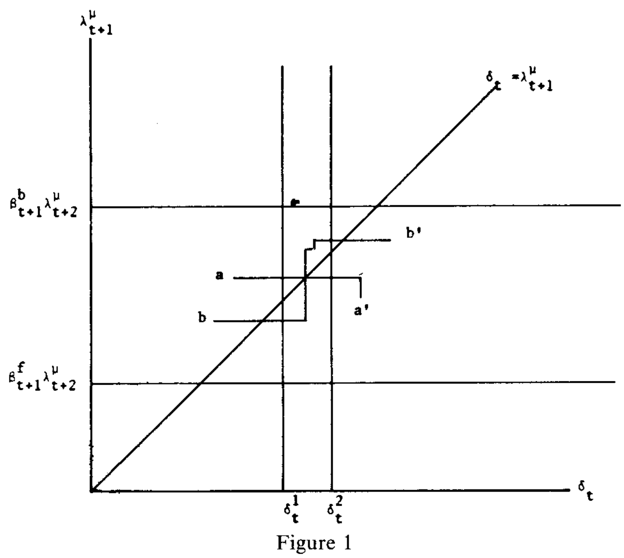

Figure 1
An Unbiased Solution

It stems from the nature of linear programming that there exists a set $\{\varepsilon_t : \varepsilon_t > 0\}$ such that the two sets of objective function coefficients $\{\delta_t\}$ and $\{\delta_t + \varepsilon_t\}$ will yield identical solutions to Problem IV. (Identical solutions are here, of course, identical in the values of the primal solution.) This feature assures that an unbiased solution can be found.**

Suppose that we partition arbitrarily the range that δ_t can take and solve all alternative linear programming problems with the different δ_t values. Suppose also that two such alternative programs yielded identical primal solutions. In Figure 1, let δ_t^1 and δ_t^2 be the alternative objective function coefficients, for a year t, associated with *the same* primal solution. The solution of the dual problem will now indicate whether or not an unbiased solution has been reached.

The graphs aa′ and bb′ represent the possible forms of variation of λ_{t+1}^μ—the coefficient of the dual solution—occasioned by variations in δ_t. In the case depicted by bb′ $\delta_t - \lambda_{t+1}^\mu$ changes sign as we go from δ_t^1 to δ_t^2 which, as seen in Figure 1, is the criterion for an unbiased solution. The reasoning for the case of aa′ is even simpler.

It is now clear that a *finite* partition of the range that the values of the δ_t's can take, can be found, such that if all combinations of these values are tried in objective functions of alternative programs, an unbiased solution will be attained. This proves the proposition.

The unbiased solution need not be unique. Perhaps in most cases the programmer will choose the solution with the highest ω—wealth at the horizon.

CONSUMPTION AS A FUNCTION OF WEALTH AT THE HORIZON

We may leave it to the interested reader to detail the formulation of Problem V—a programming model with (20) as the consumption function. We will present

* A solution consistent with (26) is any solution identical with the unbiased solution. The meaning of this will become clear shortly.

** Some difficulties may be encountered in cases of degenerate solutions. The Proposition will still hold. The proof is, however, rather tedious and will not be given here.

here only a series of remarks. (a) No misallocation problem arises since consumption is affected by final wealth only. (b) Equality of rates of transformation in consumption and production is maintained here too. (c) Wealth is taken here in the Friedman (1957) sense, and includes reproduceable as well as non-reproduceable assets of the economy. (d) No special search procedures are required in this case. However, this advantage is gained at the expense of difficulties associated with the evaluation of wealth.

CONCLUDING REMARKS

We have tried to show in this paper that dynamic capital programming models, consistent with economic theory, can practically be applied. For business analysis Problem II, with predetermined consumption outlays, can be a useful framework. Several consumption (or any other "leakage") patterns may be tried, and management will make the choice among the alternative programs.

The incorporation of consumption as an endogenous variable can be of crucial importance in the context of development. It should, however, be noted that the models discussed thus far are appropriate to the analysis of regional development problems. When it comes to national development, it is probably inappropriate to disregard demand functions for the commodities produced, as we did.

APPENDIX

SIMPLEX TABLEAU, PROBLEM IV.

Objective Function	*Dual*	0	0	0	0	δ_1	0	0	0	0	δ_2	0	0	0	0	1	*Right Hand Side Vector*
	Varia.	x_1^p	x_1^i	ϕ_1^f	ϕ_1^b	c_1	x_2^p	x_2^i	ϕ_2^f	ϕ_2^b	c_2	x_3^p	x_3^i	ϕ_3^f	ϕ_3^b	ω	
1. Finance year 1	λ_1^μ	k_1^p	k_1^i	1	-1												μ_1
2. Real input year 1	v_1	a_1^p	a_1^i														q_1
3. Consumption year 1	λ_1^c	$-\gamma z_1$		$-\gamma\rho_1^f$	$\gamma\rho_1^b$	1											α_1
4. Finance year 2	λ_2^μ	$-h_1^p$		$-h_1^f$	h_1^b		k_2^p	k_2^i	1	-1							$\mu_2-\alpha_1$
5. Real input year 2	v_2		$-b_{12}^i$				a_2^p	a_2^i									q_2
6. Consumption year 2	λ_2^c						$-\gamma z_2$		$-\gamma\rho_2^f$	$\gamma\rho_2^b$	1						α_2
7. Finance year 3	λ_3^μ						$-h_2^p$		$-h_2^f$	h_2^b		k_3^p	k_3^i	1	-1		$\mu_3-\alpha_2$
8. Real input year 3	v_3		$-b_{13}^i$					$-b_{23}^i$				a_3^p	a_3^i				q_3
9. Residual value	η		$-r_1^i$					$-r_2^i$				$-r_3^p$	$-r_3^i$	$-\beta_3^f$	β_3^b	1	Ω

NOTES TO THE TABLE

1. The tableau assumes a simple example:
   a. The programming period is three years;
   b. In every year there is one activity of every kind—current production, real investment, lending and borrowing.
   c. There is one real factor which is in limited supply in the first year and is reproducable by the real investment activity.
2. Unlike in the text, all lower case and Greek letters are scalars.
3. $h_t^p = r_t^p - \gamma z_t$
   $h_t^f = 1 + (1 - \gamma)\rho_t^f$
   $h_t^b = 1 + (1 - \gamma)\rho_t^b$

These are gross saving elements [see (2')].

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