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A Stochastic Model of Applied Research

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A mathematical model of applied research is formulated. It views applied research as a search in a given distribution; basic research shifts the distribution searched. The productivity of applied research effort is a function of the gap between technology in practice and basic knowledge. With constant basic and applied research a (stochastic) steady state emerges in which technological change is determined by the rate of progress of basic knowledge, and the technological gap by the level of applied research.

Much of research work is experimentation, and often a technological development project consists of the testing of a collection of technologies (methods, formulas, timing, varieties of crops) to find the best one. At more basic levels of research, the scope of technologies available for testing is increased. Better theoretical understanding results in superior technologies.

This experimentation process is formulated mathematically in this

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paper. The approach is illustrated mostly in terms of agricultural research but is of a wider applicability.

The approach was first suggested by the study of the development of sugarcane varieties (Evenson and Kislev 1975, chap. 3). It is inspired by Stigler's work on "The Economics of Information" (Stigler 1961) and is similar in some aspects to Nelson's treatment of R & D (Nelson 1961).[1]

Four stages were identified in the sugarcane breeding history. First, sexual reproduction of the cane plant was not known, and improvements were very slow, based on the occasional and rare cases of "natural" sexual reproduction. The second stage was the discovery of the conditions necessary to induce flowering and sexual reproduction. Seedlings were drawn from random crossings, and plants with superior potential were selected and propagated vegetatively. The third stage introduced purposeful crossing, directed to augment desired traits. The fourth stage was marked by even more specific selection programs directed to produce varieties suited to local soil, climate, and technological conditions.

Selection can be described in statistical terms as a process of random drawings from a distribution. The overall distribution in sugarcane is that of all possible genetic combinations and mutations of the species.

In the first stage of the cane breeding, the distribution sampled was that of the vegetatively reproduced, and therefore identical, plants with only occasional crossing. Thus, the drawing was limited to very few distinct observations. The population sampled was heavily concentrated around the current yield level.

In the second stage, drawing was from the distribution of all possible crossings. Compared with the first, this stage was marked by an enormous increase in the *sample variance*. The third and fourth stages were characterized by the development of techniques to affect the distributions searched, to limit it to the part of the population with the stronger desired characteristics, and thus to shift its mean.

Diminishing returns set in as search continues within the same distribution, and productivity of research increases when the search shifts to a new population. This was exemplified by the history of sugarcane varietal development in Barbados, British West Indies. Work on Stage II varieties was carried out from 1880 to 1939. The station released 10 important commercial varieties from 1902 to 1912, one from 1914 to 1928, and one important and three minor varieties from 1928 to 1939.

[1] For recent important contributions to the theory of economic search see Kohn and Shavell (1974) and Rothschild (1974). A less known early contribution is Karlin (1962) (pointed out by an anonymous *Journal* reader). Binswanger (1974) utilized a similar framework (based on an earlier draft of the present paper) in constructing his model of the micro-theory of induced innovation.

Work on Stage III varieties started in 1929. Under this project, 14 commercially important varieties were released between 1929 and 1939.

The diminishing returns to search in the same population are also demonstrated by the increasingly larger numbers of cane seedlings tested per successful "find" of a commercial variety. During the period 1929–39, the ratio of successful varieties to seedlings tested was 1 : 1,800 for the first five and 1 : 2,700 for the next nine Stage III varieties. During the same period and in the same station, this ratio was 1 : 13,000 in the older Stage II variety improvement program.

Technological research in other fields can be described in similar terms (e.g., the development and selection of chemical compounds). In our model, applied research is seen as a search within a distribution of a random variable. Basic research or learning shifts the mean of the distribution or discovers new distributions to search. An intermediate position is occupied by work aimed at increasing the variance of the samples. The model formulated is an economic one; it is assumed that the objective function of the system is to maximize the expected present value of future income, inclusive of research cost. The system modeled can be either an economy with a research sector, or limited section of the economy with a research team working to improve technology within the section.

The main innovation of the study is in the treatment of applied research and testing, emphasis in the following discussion being given to this aspect of research work. The framework of the analysis is initially to set up a pure-search model and later to introduce the possibility of more basic research into the model.

Applied Research (Testing)

The scientist (the scientific team) is assumed, in this section, to be presented with a given distribution of outcomes whose parameters he cannot directly affect; his work is strictly testing; no basic research is done. To be concrete, imagine a research project aimed at increasing the yield of a crop. To simplify, assume that net income is in direct proportion to yield. Work on the project is composed of a succession of experiments. In control theoretic language, the state of the system is the yield at any point in time, and the results of the experiments are the transition equations—changing the yield level. The control variable is the extent of experimentation at any stage or time period. Again for simplicity, assume that the only control variable is the number of trials in an experiment—the number of drawings from a random distribution.

Since the transition equation is a random process, the state variable is random, but other sources of randomness and uncertainty, such as weather effects, are disregarded.

At any period t an experiment composed of n_t trials is conducted. The following variables are defined as follows:

Y_t: yield, technology level, net income in time t ($t = 0, 1, 2, \ldots$), the *state* variable;
n: number of trials, $n = 0, 1, 2, \ldots$, the *control* variable;
$c(n)$: cost of experimentation, with $c(0) = 0$, and assume that $c(n)$ is increasing with n at an increasing rate;
x_i: yield in trial i, ($i = 1, 2, \ldots, n$);
$f(x)$: probability density function of x;
$F(x)$: cumulative distribution of x;
z: the largest value in a sample of the random variate x;
$\alpha = \frac{1}{1 + r}$: the discount factor with r the rate of interest;
V: the objective function;
E: expectation operator;
$\Delta y = y_t - y_{t-1}$: yield increment;
$E\{x\}$: mean of x;
Var (x): variance of x;
D: first difference operator, e.g., $D_n c \equiv c(n) - c(n - 1)$.

The search process is a sequence of *experiments*, each composed of n_t *trials*. A single trial can be a test of a technique—one variety of a crop, a certain dose of fertilizers, one planting date. Because of the variability in experimental conditions, a trial is usually carried out in a number of replicas. This variability is, however, disregarded here, and it is assumed that a trial has a single outcome—an observation from the distribution of yields, in our example.

Each trial results in an observation—one drawing from a random population. The outcome of the experiment is the best observation in the sample. The statistical process of choosing the best outcome from a set of random drawings is treated under the heading of the theory of extreme values (Gumbel 1958; Epstein 1960) in the general subject of order statistics.

Utilizing the symbols introduced earlier, x_i is the yield in trial i and $z = x_j$, $x_j \geq x_i$ ($i = 1, 2, \ldots, n$).

The cumulative distribution of z is

$$H_n(z) = \Pr(\text{all } x_i \leq z) = F^n(z), \tag{1}$$

and the density function (if existing) is

$$h_n(z) = nF^{n-1}(z) f(z). \tag{2}$$

The analysis will be illustrated in terms of the exponential distribution:

$$f(x) = \lambda e^{-\lambda(x-\theta)}, \quad \theta \leq x; \tag{3}$$

$$F(x) = 1 - e^{-\lambda(x-\theta)}; \tag{4}$$

$$E(x) = \theta + \frac{1}{\lambda}; \tag{5}$$

$$\mathrm{Var}\ (x) = \frac{1}{\lambda^2}. \tag{6}$$

The cumulative distribution of the largest values is, employing (1),

$$H_n(z) = [1 - e^{-\lambda(z-\theta)}]^n, \tag{1'}$$

and the probability density function is

$$h_n(z) = \lambda n[1 - e^{-\lambda(z-\theta)}]^{n-1} e^{-\lambda(z-\theta)}. \tag{2'}$$

See figure 1 where $\lambda = 1; \theta = 0; n = 1, 2, \ldots, 5$.

The expected value and the variance of z are (Gumbel 1958)

$$E_n(z) = \theta + \frac{1}{\lambda} \sum_{i=1}^{n} \frac{1}{i}; \tag{7}$$

$$\mathrm{Var}_n\ (z) = \frac{1}{\lambda^2} \sum_{i=1}^{n} \frac{1}{i^2}. \tag{8}$$

In each time period, an experiment consisting of n trials (n drawings from a random distribution) is conducted. The result of each trial is an observation, x, the yield level associated with the tested technique. If the outcome of the experiment is higher than y, the yield under the current technique, the new technique is put to use and y increases. If not, y does not change. The search is then repeated, within the same distribution, perhaps with a different n.

Formally,

$$\Delta y = z - y \quad \text{if } y < z \tag{9}$$

$$\Delta y = 0 \qquad \text{otherwise.}$$

The expected value of the technology increment is[2]

$$E_n(\Delta y) = \int_y^\infty (z - y) h_n(z)\, dz = \int_y^\infty [1 - F^n(z)]\, dz. \tag{10}$$

[2] Strictly speaking, the last integral in (10) exists for F such that $F^n > 1 - z^{-2}$ for large values of z, which holds for the exponential distribution. For a more general treatment of this case, see De Groot (1970, p. 246). We are indebted to a *Journal* reader for this reference.

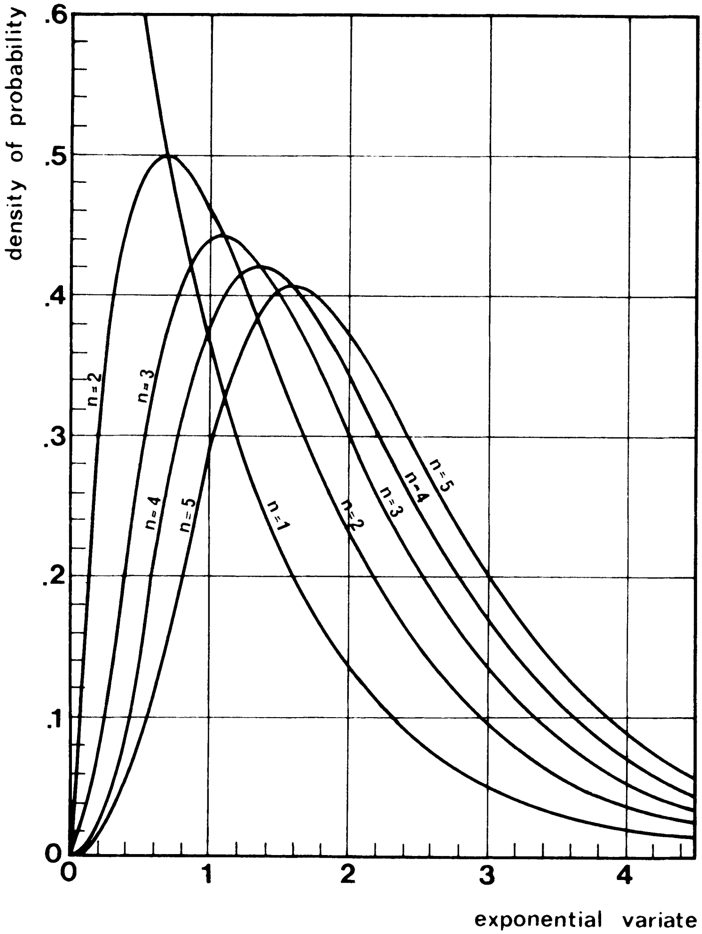

Fig. 1.—The function $h_n(z) = n(1 - e^{-x})^{n-1}e^{-x}$. Source: Gumbel 1958

As expected, the contribution of an additional trial is positive and diminishing:

$$E_n(\Delta y) - E_{n-1}(\Delta y) = \int_y^{\infty} F^{n-1}(z)[1 - F(z)]\, dz. \tag{11}$$

Since $0 < F(z) < 1$ for $y < z < \infty$, the difference $E_n(\Delta y) - E_{n-1}(\Delta y)$ is positive and is a decreasing function of n.

For the exponential distribution the expected value of Δy is

$$E_n(\Delta y) = \sum_{i=1}^{n} \frac{1 - [1 - e^{-\lambda(y-\theta)}]^i}{\lambda i}, \tag{12}$$

and its variance

$$\mathrm{Var}_n\,(\Delta y) = \sum_{i=1}^{n} \frac{2\mathrm{E}_i(\Delta y)}{\lambda i} - E_n^2(\Delta y). \qquad (13)$$

Time Evolution and Optimization

The search process proceeds over time, improving technology whenever possible. The present value of the system of production and technological research is

$$V(y) = E\left\{\sum_{t=0}^{\infty} [y_t - c(n_t)]\alpha^t\right\}, \qquad (14)$$

where y is the present technology level (at $t = 0$). Let $V^*(y)$ denote the value of a system when an *optimal* research policy is followed; it can be written in the form of a recursion functional as

$$V^*(y) = \max_n \left[y - c(n) + \alpha \int_y^{\infty} V^*(z)\, dF^n(z) + \alpha F^n(y) V^*(y) \right]. \qquad (15)$$

The last two terms on the right-hand side of (15) are the benefits $B(y, n)$ due to the present experiment

$$B(y, n) = \alpha \int_y^{\infty} V^*(z)\, dF^n(z) + \alpha F^n(y) V^*(y),$$

and they are, respectively, the expected value of a system starting next period from a better than current technology—if such a technology is found—and the value of the current system weighted by the probability that the outcome of the experiment will not exceed the current yield.

Incremental benefits from increasing the number of trials in an experiment are positive and diminishing:

$$\begin{aligned} D_n B &\equiv B(y, n) - B(y, n-1) \\ &= \alpha \int_y^{\infty} V^*(z)[nfF^{n-1} - (n-1)fF^{n-2}]\, dz \qquad (16) \\ &\quad + \alpha V^*(y)(F^n - F^{n-1}). \end{aligned}$$

Integrating by parts, we get

$$D_n B = \alpha \int_y^{\infty} \frac{\partial V^*}{\partial z} (F^{n-1} - F^n)\, dz \geq 0, \qquad (17)$$

since $(\partial V^*/\partial z) \geq 0$, as a discovery of a better technology can only improve the value of the system. This proves that incremental benefits are positive; they are diminishing;

$$D_nB - D_{n-1}B = \alpha \int_y^\infty \frac{\partial V^*}{\partial z} (2F^{n-1} - F^n - F^{n-2})\, dz \leq 0, \quad (18)$$

since $2F^{n-1} \leq F^n + F^{n-2}$.

The cost function $c(n)$ is, by assumption, increasing with n at an increasing rate:

$$D_nc \equiv c(n) - c(n-1) > 0; \quad (19)$$

$$D_nc - D_{n-1}c > 0.$$

Since incremental returns are decreasing, optimal n will be such that

$$|D_{n-1}c - D_{n-1}B| \geq |D_nc - D_nB| \leq |D_{n+1}c - D_{n+1}B|. \quad (20)$$

This is illustrated in figure 2.[3]

Economic Properties of Optimal Solutions

1. The optimal number of experiments is a decreasing function of the rate of interest, as D_nB in (16) decreases with r.

Since the extent of experimentation, technological research, is a decreasing function of the rate of interest, technological progress will also be a decreasing function of the rate of interest. This is one aspect in which the present model resembles Solow's embodiment model (Solow 1960). In our model, technical progress is the outcome of investment in research; in Solow's model it is embodied in new capital assets. In both cases, the rate of progress is a function of the rate of interest.

2. It is useful to view technological research as filling a gap between basic knowledge and the level of technology in practice (Nelson and Phelps 1966). In the exponential distribution (3), the level of basic research can be represented by the parameter θ. The *smaller* the difference $y - \theta$, the *larger* the technological gap and, in our model, the easier it is to improve the technology. The analytical difficulty here is that a change in θ modifies the distribution searched and affects $V^*(y)$ by changing all

[3] The present optimization framework can undoubtedly be expanded to include adaptive search, sequential experimentation with various stopping rules, and similar cases. This has not been done here in order to preserve simplicity and shortness. The qualitative results and the major characteristics of research systems focused upon in the present model would have remained unchanged. Note in this respect Rothschild's (1974, p. 689) conclusion that "not invariably, but in many instances, the qualitative properties of the optimal-search strategies . . . are the same [in the adaptive case] as in the simpler case when the distribution is assumed known."

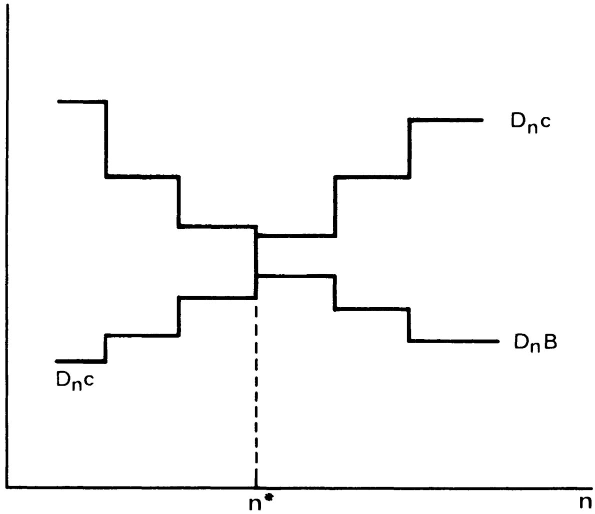

FIG. 2.—Optimal n (n*)

future (expected) steps. We therefore show explicitly the effect of a change in θ on the expected value of a system in which *only one* experiment is conducted. The benefits of such a system are

$$B^1(y, n) = \frac{\alpha}{r}\int_y^{\infty} znf(z)F^n(z)\, dz + \frac{\alpha}{r}F^n(y)y, \tag{21}$$

and in the exponential distribution [with the density function $\lambda e^{-\lambda(z-\theta)}$]

$$\frac{\partial B}{\partial \theta} = -\frac{\partial B}{\partial z},$$

therefore

$$\frac{\partial B^1}{\partial \theta} = -\frac{\alpha}{r}\int_y^{\infty} z\frac{\partial^2 F^n}{\partial z^2}\, dz - \frac{\alpha}{r}\frac{\partial F^n}{\partial z}y.$$

Integrating by parts

$$\frac{\partial B^1}{\partial \theta} = \frac{\alpha}{r}[1 - F^n(y)] \geq 0. \tag{22}$$

Since (22) is true for any value of y, current and future, an increase in θ will increase the value of $B(y, n)$ for any unrestricted system.

3. Economies of scale can be introduced by multiplying V in (14) and (15) by a scale factor. The larger this factor, the higher optimal n and the faster technological progress.

This formulation will represent cases in which technology is identical throughout the volume of operation of a firm or an economy, for example, if a new variety will raise yield equally in all fields or if a new chemical compound will increase productivity equally in all its applications. In other cases, however, scale represents, at least partially, variability in the conditions of production: the larger the area sown to a crop, the more variable growing conditions are; and this variability factor dampens the direct, proportional scale effect.

4. The benefits are a decreasing function of y, since y is the lower bound of the integral in $B(y, n)$. So also optimal number of trials, n, is a decreasing function of the level of technology, y. Eventually optimal $n = 0$ and technological research will stop. This will be the case when

$$D_n B \leq c(n) \tag{23}$$

for any n. Particularly,

$$D_1 B \leq c(1), \tag{24}$$

or, written explicitly,

$$\alpha \int_y^\infty V^*(z) f(z)\, dz + \alpha F(y) V^*(y) \leq c(1). \tag{25}$$

Thus after a certain level, technology will stagnate forever. This stagnation will not occur if basic science progresses continuously. This possibility is analyzed in the second part of the paper.

5. The inequality in (25) is not only the condition for stopping a research project, it is also the condition for not starting one. A project will not be undertaken if the expected benefits of even a single trial in a single experiment will not exceed the corresponding cost:

$$\frac{\alpha}{r} E_1(\Delta y) = \frac{\alpha}{r}\frac{1}{\lambda}[1 - 1 - e^{-\lambda(y-\theta)}]$$

$$= \frac{\alpha}{r}\frac{1}{\lambda} e^{-\lambda(y-\theta)} \leq c(1). \tag{26}$$

If the system starts from a position represented by (26), then a change in one of the following three parameters can make technological research justifiable: (*a*) A reduction in the rate of interest; (*b*) An increase in the basic knowledge parameter θ, since

$$\frac{\partial E_1(\Delta y)}{\partial \theta} = e^{-\lambda(y-\theta)} \geq 0;$$

(*c*) A decrease in λ, since

$$\frac{\partial E_1(\Delta y)}{\partial \lambda} = -\frac{1}{\lambda} e^{-\lambda(y-\theta)} \left(\frac{1}{\lambda} + y - \theta\right) < 0.$$

The last is an increase in variance of the population searched. This is an example of variance increasing research; for instance, the International Rice Research Institute in the Philippines accumulated a collection of 15,000 rice varieties increasing substantially the variance of the rice varieties population searched for genetic material.

Basic Research

In terms of the present model, basic research can be classified into three categories: basic research can shift the mean of the distribution searched, it can change its variance, or it can create new technologies—discover new distributions to search. The following discussion will analyze the first case; the last is treated in the concluding section of the paper.[4]

With *mean shifting basic research*, θ is no longer constant. Its per period growth, due to basic research, is $\Delta\theta$, and *it is assumed throughout the discussion that $\Delta\theta$ is constant.* It is shown below that optimal n converges, if $\Delta\theta$ is constant, to a constant level. But a constant level of applied research will not be limited to optimal system. Quite often a research organization is operated at a certain constant level determined by budgets and political circumstances. In such a case an interesting property emerges.

Property (steady state).—If both basic research and technological research proceed at constant rates, a stochastic steady state emerges at which technology in practice improves at a rate equal to the rate of advancement of basic research. Formally, if $\Delta\theta$ = constant, n = constant, then eventually $E_n(\Delta y) = \Delta\theta$.

Proof.—For the analysis of the search process, the state variable is not y but, rather, $\xi = y - \theta$, and we have to show that, if n = constant, $\Delta\theta$ = constant, then ξ = constant.

Rewrite $E_n(\Delta y)$ as

$$E_n(\Delta y) = \sum_{i=1}^{n} \frac{1 - (1 - e^{-\lambda\xi})^i}{\lambda i};$$

now

$$\frac{\partial E_n(\Delta y)}{\partial \xi} = -\sum_{i=1}^{n} e^{-\lambda\xi}(1 - e^{-\lambda\xi})^{i-1}. \tag{27}$$

As ξ grows, $E_n(\Delta y)$ decreases, and vice versa. There exists a steady-state value of ξ, ξ^*, a function of n and $\Delta\theta$, which will be maintained (stochastically); it is defined by

$$E_n(\Delta\xi^*) = \sum_{i=1}^{n} \frac{1 - (1 - e^{-\lambda\xi^*})^i}{\lambda i} - \Delta\theta = 0. \tag{28}$$

Q.E.D

[4] The case of variance increasing research is discussed, in a similar framework, in Evenson and Kislev (1975, chap. 8).

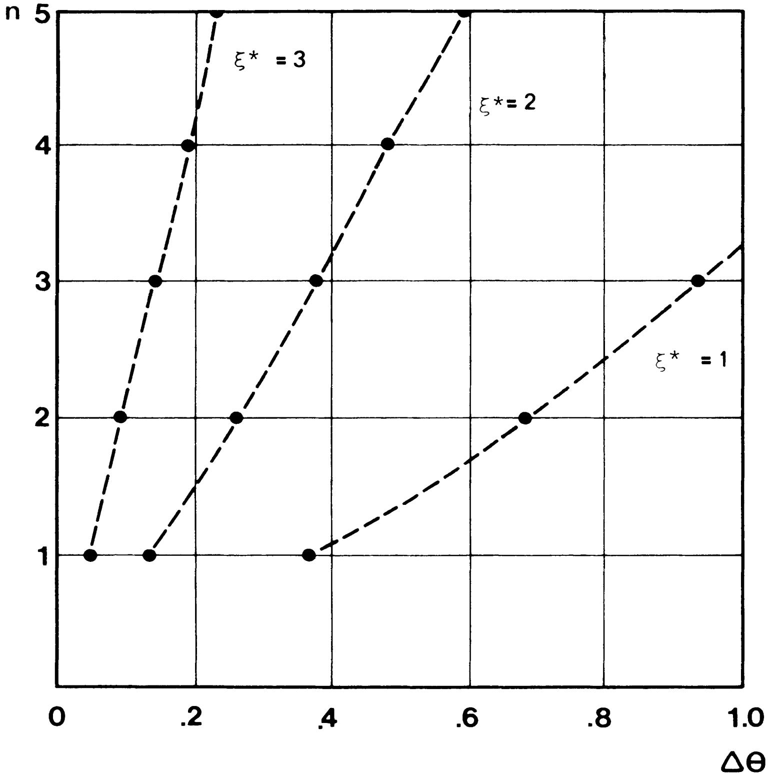

FIG. 3.—Steady-state ξ for $\lambda = 1$; $n = 1, 2, \ldots, 5$; $0 < \Delta\theta < 1$

Steady-state ξ can be written as

$$\xi^* = \xi^*(n, \Delta\theta). \tag{29}$$

Figure 3 illustrates equation (29) for $\xi^* = 1, 2, 3$; $\lambda = 1$; $n = 1, 2, \ldots, 5$; $0 < \Delta\theta < 1$. The broken lines in the figure only connect steady-state $(n, \Delta\theta)$ pairs; other points on the lines are meaningless. We shall now investigate algebraically the properties of (29).

For a given n, steady-state ξ is a decreasing function of $\Delta\theta$. To see this, differentiate (28) with respect to $\Delta\theta$

$$\frac{\partial E_n(\Delta\xi^*)}{\partial\Delta\theta} = -\frac{\partial\xi^*}{\partial\Delta\theta}\sum_{i=1}^{n}(1 - e^{-\lambda\xi^*})^{i-1}e^{-\lambda\xi^*} - 1 = 0; \tag{30}$$

$$\frac{\partial\xi^*}{\partial\Delta\theta} = -\left[\sum_{i=1}^{n}(1 - e^{-\lambda\xi^*})^{i-1}e^{-\lambda\xi^*}\right]^{-1}. \tag{31}$$

The interpretation of this result is that in the steady state for a given level of technological research (a given n), the higher the rate of basic knowledge, the larger the gap between basic knowledge and technology in practice (the smaller ξ^*). This property stems from the fact that the bigger the gap, the more productive will technological research be. In the steady state this gap, represented by the variable ξ^*, is the only endogenous variable in the system; the bigger $\Delta\theta$, the larger the gap which will maintain $E_n(\Delta y) = \Delta\theta$.

Steady-state ξ can even become negative. For example, if $\lambda = 1$ and $n = 1$, $\xi^* < 0$ for $\Delta\theta < 1.0$. This can readily be seen by rewriting (28) for $\Delta\theta = 1$:

$$E_1(\Delta\xi^*) = e^{-\xi^*} - 1 = 0. \tag{28'}$$

Clearly, $\xi^* = 0$ in (28′) and by (31) $\xi^* < 0$ for $\Delta\theta > 1$.

A negative ξ^* means that basic knowledge has advanced, relative to technology in practice, to such a level that *any search* will find (with probability one) superior technologies. It should not be surprising that such cases are not observed often in the real world. Technological research is an economic activity, perhaps not always conducted optimally, but surely directed into areas with obvious and safe gains and, thereby, eliminating these potential gains.

To find the behavior of ξ^* as a function of n, approximate the sum in (28) by an integral

$$E_n(\Delta\xi^*) = \int_0^n \frac{1 - (1 - e^{-\lambda\xi^*})^i}{\lambda i}\, di - \Delta\theta = 0. \tag{28''}$$

Differentiating (28″) with respect to n and rearranging terms, we get

$$\frac{\partial\xi^*}{\partial n} = \frac{1 - (1 - e^{-\lambda\xi^*})^n}{n\lambda e^{-\lambda\xi^*} \int_0^n (1 - e^{-\lambda\xi^*})^{i-1}\, di} > 0. \tag{32}$$

The steady-state ξ grows with n—the more technological research is conducted, the smaller the steady-state gap between technology and basic knowledge.

To find the optimal level of experimentation, note that the present value of the system in a steady state is

$$\begin{aligned}\omega &= \sum_{t=0}^{\infty} [E(y_t) - c(n)]\alpha^t \\ &= \frac{y_0}{r} + \sum_{t=0}^{\infty}\left[\sum_{i=0}^{t} E_n(\Delta y)\right]\alpha^t - \frac{c(n)}{r} \\ &= \frac{\theta_0 + \xi^* - c(n)}{r} + \Delta\theta \sum_{t=0}^{\infty} t\alpha^t. \end{aligned} \tag{33}$$

The only terms in the last line of (33) which are functions of n are ξ^* and $c(n)$. Therefore, the problem of optimal n can be formulated, disregarding the other terms, as

$$\underset{n}{\text{Max}}\, [\xi^*(n, \Delta\theta) - c(n)]. \tag{34}$$

The maximization in (34) is not limited to $t = 0$, as one may erroneously conclude from an examination of (33); any t can be viewed as $t = 0$. The problem is to find what constant level of technological research to maintain so as to reach an optimal (stochastically) constant level of ξ^*. In other words, in the steady-state situation, technological research does not determine the *rate of progress* of technology—this rate is equal to the exogenously determined rate of progress of basic knowledge—rather, technological research determines the *level* of technology in practice or, rather, the constant technological gap.[5] In this respect, the present model is similar in character to the Nelson and Phelps (1966) model, where in a steady-state equilibrium schooling determines the technological gap, while the rate of technical progress is determined by the rate of advancement of basic science.

From (32) it can be shown that

$$\frac{\partial^2 \xi^*}{\partial n^2} < 0, \tag{35}$$

and since $D_n c \equiv c(n) - c(n - 1) \geq 0$, optimal n will be the value for which

$$|D_{n-1}\xi^* - D_{n-1}c| \geq |D_n\xi^* - D_n c| \leq |D_{n+1}\xi^* - D_{n+1}c|, \tag{36}$$

where $D_n\xi^* \equiv \xi^*(n, \Delta\theta) - \xi^*(n - 1, \Delta\theta)$.

What effect will the rate of basic research have on optimal n in the steady state? On the one hand, the faster the advancement of basic knowledge, the higher the productivity of applied research; on the other hand, in the steady state, the higher the level of basic research, the smaller the technological gap. In fact, as can be seen by differentiating (32) with respect to $\Delta\theta$, the sign of $\partial^2\xi^*/\partial n\, \partial\Delta\theta$ is indeterminate, and the magnitude of this second derivate is very small. In the numerical example used in the illustration of figure 3, optimal n, over the range $0 < \Delta\theta < 1$, is almost always independent of $\Delta\theta$. This can be realized by examining figure 4, where $D_n\xi^*$ is plotted against n for $\Delta\theta = .1, .5, .9$. Note, for example, that if $c(n) = .4n$, ($D_n c = .4$), optimal n will be $n = 3$ for any $\Delta\theta$ at least over the range $.1 \leq \Delta\theta \leq .9$.

This tendency of the applied research to be a decreasing (at least, not

[5] An exception is the case of $n = 0$—no technological research—in which technology in practice is stagnant and the technological gap constantly increases as basic research adds to the stock of basic knowledge.

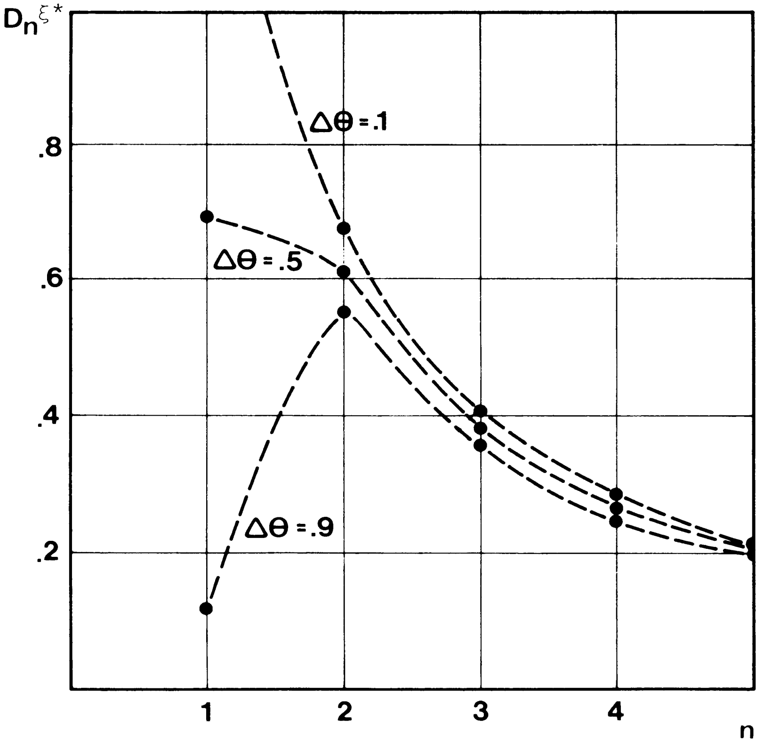

FIG. 4.—$D_n\xi^* \equiv \xi^*(n, \Delta\theta) - \xi^*(n-1, \Delta\theta)$ for $\lambda = 1, n = 1, 2, \ldots, 5$

an increasing) function of the rate of basic scientific advancement is a reflection of the steady-state conditions, where the faster basic knowledge accumulates, the larger the technological gap (the smaller ξ^*) and the smaller may be the optimal n. A different case is discussed in the following section.

Exogenously given $\Delta\theta$ and the cost function $c(n)$ determine the pair optimal n and ξ^* in the steady state. If the system starts with $\xi < \xi^*$, then marginal returns to experimentation will be larger than in the steady state and optimal n will tend to be larger than steady-state optimal n. With time, as $\xi \rightarrow \xi^*$, optimal n will approach its steady-state value. A similar convergence to equilibrium, in the opposite direction, will take place if initially $\xi > \xi^*$. The steady state is the long-run equilibrium stable position of an optimal system.

Concluding Remarks on the Discovery of New Technologies

Science creates a great variety of applicable technologies, from electricity to antibiotics, from atomic energy to genetic engineering. In terms of the present model, the discovery of new technologies is represented by the opening up of new exponential distributions for search. For simplicity, assume that all technologies are characterized by identical distributional attributes: identical λ and θ parameters; moreover, assume that $\theta = 0$ in all distributions. These are strong assumptions, but they permit focusing the discussion on some essential issues. The relaxation of these assumptions will complicate the analysis greatly and will have to be postponed to another occasion.

Let k be the technology index; k is a vintage parameter: The more recent the technology, the higher its k value. The expected income from an experiment in technology k is

$$E_{n(k)}[\Delta y(k)] = \sum_{i=1}^{n(k)} \frac{1 - [1 - e^{-\lambda \xi(k)}]^i}{\lambda i}. \tag{37}$$

For newly discovered technologies, (37) reduces to

$$E_{n(k)}[\Delta y(k)] = \sum_{i=1}^{n(k)} \frac{1}{\lambda i}, \tag{37'}$$

since $y = \theta$ for new technologies.

At any point in time there exists a spectrum of technologies in the economy. Technological research is now an industry with a rising supply function

$$c(n),\ n = \sum_k n(k),$$

of factors of research. Experiments will be allocated to technologies according to their comparative standing in the current technological spectrum.

The discussion that led to equation (20) can now be repeated, and again that equation is the condition for optimal allocation of experiments with one departure from the previous analysis. If there is a large number of distribution to search, then $D_n c$ is now the incremental cost function for the industry; it is taken as datum when considering a single distribution. In other words, with a very large number of technologies, the allocation of research efforts becomes analogous to the organization of a competitive industry with separate firms doing research in the separate distributions.

In addition to ordering distributions by their vintage index k, they can now be ordered by their $y(k)$ value. There will be a strong (negative) correlation between y and the vintage index, but these will not be the same orderings. A break-even value of y will be y^* maintaining

$$\frac{\alpha}{r} \frac{1}{\lambda} e^{-y^*} = c(1). \tag{38}$$

The term on the left in (38) is identical to the present value of the incremental income due to an experiment of one trial. Distributions with $y > y^*$ will not be subject to search any more. These distributions are *technologically exhausted.* Exhaustion is an economic phenomenon; a reduction in the cost function, for example, will cause the resumption of search in previously exhausted technologies.

Technological exhaustion should be carefully distinguished from *obsolescence.* The last is a market phenomenon, reflecting changing comparative advantages of technologies and not treated in the present study.

The rate of progress of basic science is measured in the model by the number of new technologies discovered per time period. The larger this number, the higher the demand for research factors and the higher the rate of experimentation. At the same time, the higher the rate of advancement of basic science, the faster the rate of exhaustion of old technologies. In fact, in a steady state, the number of exhausted technologies will be equal to the number of newly discovered technologies, and the larger this number, the *less* search will be conducted in a typical technology before its exhaustion.

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